

9. MATH 30A, HOMEWORK 9

Due: Thursday, Nov 5.

9.1. Problems from section 15. p. 151: # 14, 31, 35, 38

Here are the answer to # 13, #37

In class I proved that the center of S_3 is $\{e\}$ and I stated without proof that the commutator subgroup of S_n is A_n for all n . So, you can use those facts.

#13. The center of D_4 is $\langle t^2 \rangle = \{e, t^2\}$. The commutator subgroup of D_4 is also $\langle t^2 \rangle$. Proof: All commutators of two elements in D_4 are either e or t^2 . You can do this systematically as follows.

- (1) $[t^i, t^j] = e$ since any two powers of the same element commute.
- (2) First we need to remember that st^n is a reflection for $n = 1, 2, 3, 0$. So, $(st^n)^{-1} = st^n = t^{4-n}s$. (The second is from the equation $(ab)^{-1} = b^{-1}a^{-1}$). So,

$$[st^n, t^m] = st^n t^m st^n t^{4-m} = s \underbrace{t^{n+m}}_{st^{4-n-m}} t^n t^{4-m} = s^2 t^{4-n-m+n+4-m} = t^{2m}$$

and $t^{2m} = e$ or t^2 depending on the parity of m .

- (3) $[st^n, t^m] = st^n t^m st^n t^{4-m} = t^{4-n-m+n+4-m} = t^{2m}$ which is also either e or t^2 .
- (4) $[t^m, st^n] = [st^n, t^m]^{-1} = t^{2m} = e$ or t^2 .

#37. Show that, if G is nonabelian then the quotient group $G/Z(G)$ cannot be cyclic.

Proof: (By contradiction) Suppose that $G/Z(G)$ is cyclic and generated by the element aZ (where $Z = Z(G)$). Then the elements of G/Z are: $a^n Z$ for integers n . But G is a union of these cosets. So, $G = \bigcup a^n Z$. This means that every element of G can be written in the form $a^n z$ where n is an integer and $z \in Z$. So, take two elements of G . They would be $a^n z_1, a^m z_2$. But z_1, z_2 are elements of $Z(G)$ so they commute with every element of G :

$$(a^n z_1)(a^m z_2) = a^n a^m z_1 z_2 = a^{n+m} z_2 z_1 = a^m a^n z_2 z_1 = (a^m z_2)(a^n z_1)$$

So, any two elements of G commute. So, G is abelian which is a contradiction proving the theorem.

9.2. Problems from section 18. p. 174 # 12, 16, 18, 24,

Here are the answers to # 11, # 15, 25:

11. Is the following a ring with unity and is it a field?

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

Answer: This is a subset of a known field \mathbb{R} . Therefore, to verify that it is a commutative ring with unity it suffices to check that it contains 0 and 1 and is closed under the operations of $+$, $-$, \times .

(1) $0 = 0 + 0\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$.

(2) $\mathbb{Z}[\sqrt{2}]$ is closed under addition:

$$(a + b\sqrt{2}) + (x + y\sqrt{2}) = (a + x) + (b + y)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

(3) $\mathbb{Z}[\sqrt{2}]$ is closed under additive inverse (negation):

$$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

(4) $\mathbb{Z}[\sqrt{2}]$ is closed under multiplication:

$$(a + b\sqrt{2})(x + y\sqrt{2}) = (ax + 2by) + (ay + bx)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

(5) $\mathbb{Z}[\sqrt{2}]$ contains unity: $1 + 0\sqrt{2}$.

We can conclude that $\mathbb{Z}[\sqrt{2}]$ is a subring of \mathbb{R} with 1 and therefore an integral domain. $\mathbb{Z}[\sqrt{2}]$ is not a field since not all nonzero elements have inverses. For example $\sqrt{2}$ does not have an inverse in $\mathbb{Z}[\sqrt{2}]$ since

$$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \notin \mathbb{Z}[\sqrt{2}].$$

(The coefficient of $\sqrt{2}$ is not an integer.)

#15. The ring $\mathbb{Z} \times \mathbb{Z}$ has unity $(1, 1)$. The units are solution of the equation $(a, b)(c, d) = (1, 1)$ which gives $(ac, bd) = (1, 1)$. So, a, b, c, d must be ± 1 giving four units:

$$(1, 1), (-1, 1), (1, -1), (-1, -1)$$

#25. If $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ is a ring homomorphism then:

$$\phi(a, b) = \phi(a \cdot (1, 0) + b \cdot (0, 1)) = a \cdot \phi(1, 0) + b \cdot \phi(0, 1)$$

So, ϕ is completely determined by the numbers $\phi(1, 0)$ and $\phi(0, 1)$. Now use the multiplicative properties: Both of these elements of $\mathbb{Z} \times \mathbb{Z}$ is idempotent: $(1, 0)^2 = (1, 0)$ and $(0, 1)^2 = (0, 1)$. So, $\phi(1, 0)$ and $\phi(0, 1)$ are idempotents in \mathbb{Z} . So, they are either 1 or 0 (the only integer solutions of $x^2 = x$). One more thing:

$$(1, 0)(0, 1) = (0, 0)$$

So, $\phi(1, 0)\phi(0, 1) = \phi(0, 0) = 0$. So one of these numbers must be zero. So there are exactly three possibilities.

- (1) Both $\phi(1, 0), \phi(0, 1) = 0$ making $\phi(a, b) = 0$ for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$
- (2) $\phi(1, 0) = 1$ and $\phi(0, 1) = 0$ making $\phi(a, b) = a$
- (3) $\phi(a, b) = b$.