1. Math 30a, Fall 2009
Answers to **Quiz 1**

1.1. Explain why each of the following is NOT a group.

(1) $(\mathbb{Q}, \cdot)$ (rational numbers under multiplication) 0=0/1 does not have a multiplicative inverse.

(2) The set of positive real numbers under the operation $a * b = e^{ab}$. This operation does not have a unit. If E were the unit we would have $E * 1 = 1$ which is impossible since $e^{ab} > 1$ for any positive a, b. It also is not associative, but one reason is sufficient.

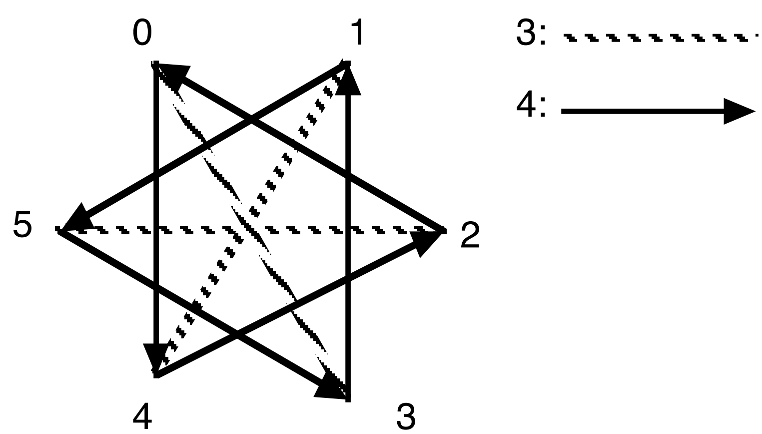
(3) The subset H of $GL(2, \mathbb{R})$ given by the equation:

$$H := \{A \in GL(2, \mathbb{R}) \mid \det A \in \mathbb{Z}^+\}$$

with operation given by matrix multiplication.

This set does not have the inverse of its elements. For example $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is in the set since it has determinant $\det A = 2$. But the inverse $A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ has determinant 1/2. So, $A^{-1} \notin H$.

1.2. Draw the Cayley digraph for the group $\mathbb{Z}_6$ with generating set $\{3, 4\}$. Make sure to label each vertex. Use your diagram to explain something about the group.



The diagram shows that $3, 4$ together generate the group since the graph is connected. However, neither of these elements will be itself generate the group $\mathbb{Z}_6$. This is shown in the graph by the fact that the dotted lines only connect 1 to 4, 0 to 3 and 2 to 5 but you cannot, e.g., get from 0 to 5 following dotted lines. Similarly, the solid lines connect the even numbers together and the odd numbers but you cannot get from an even number to an odd number following just the solid lines.

1.3. Find all cyclic subgroups of the symmetric group on 3 letters. How many are there? Each of the 6 elements of the symmetric group generate a cyclic subgroup. However, two of those elements give the same subgroup.

$$\langle (123) \rangle = \{e, (123), (132)\} = \langle (132) \rangle$$

So there are only 5 cyclic subgroups of S_3.

1.4. Give an example of an infinite group G with a subgroup H having 4 elements. Some correct answers are: $G = U = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $H = U_4 = \{1, -1, i, -i\}$, $G = GL(2, \mathbb{R})$ and

$$H = \left\{ I_2, -I_2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

The set of all binary number where the addition is coordinatewise (without carrying) is an infinite group with subgroup $H = \{0, 1, 10, 11\}$

Some popular incorrect answers were: $G = U_n, H = U_4$. Here U_n has only n elements. So, it is not infinite. Also, U_n will contain U_4 only if n is a multiple of e. Another wrong answer is $G = \mathbb{Z}$ and $H = \mathbb{Z}_4$. This is not a subgroup of $\mathbb{Z}$ since subgroups need to use the same operation as the larger group. The operations $+$ and $+_4$ are not the same since $2 + 2 = 4$ and $2 +_4 2 = 0$.

1.5. Prove that a group in which every element (except the identity) has order 2 is abelian. This was a homework problem.

(1) First restate the problem using the definitions and with some choice of notation. What does this question ask for? Given that $g^2 = e$ for all $g \in G$, we need to show that $ab = ba$ for all $a, b \in G$. "commutative" is a synonym and does not explain what "abelian" mean.
(2) Make sure to justify each step in your proof or calculation.