1. Math 30a, Fall 2009
Review for Quiz 1

Rules for the quiz: Closed book. 40 minutes. *You may bring notes written on a single piece of paper (both sides, letter size). You need to decide what should be written on your paper. You can't ask me about definitions. I am going to ask you!*

1.1. List of topics.

Binary operation Find an infinite number of binary operations on $\mathbb{Z}$.

Isomorphism Find an isomorphism from $(\mathbb{Z}_2, +_2)$ to $GL(1, \mathbb{Z})$

Definition of group (Identity, inverse, associativity) Show that the set of all positive rational numbers a/b where a, b are positive odd integers is a multiplicative group.

Abelian (commutative) groups Give an example of a noncommutative finite group.

Subgroups Show that the set of rational number a/b where b is an odd integer is a subgroup of the group $(\mathbb{Q}, +)$.

Determinant $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ given by a condition on the determinant, namely $\det A = 1$. Find other conditions on the determinant which will give subgroups of $GL(n, \mathbb{R})$.

Cyclic groups $\mathbb{Z}_n$, U_n Find all subgroups of $\mathbb{Z}$. Find all isomorphisms $\mathbb{Z}_5 \to U_5$. (How many are there?)

Division algorithm $(k = qn + r)$ State the theorem.

Order of groups and elements. Does a group of order n always have an element of order n

Generators The group $(\mathbb{Z}^2, +)$ of all points in the plane with integer coordinates is generated by two elements $(1, 0)$ and $(0, 1)$. Find one billion more pairs of generators.

Cayley digraphs Draw the Cayley digraph for the group $\mathbb{Z}_5$ with generator 2.

Permutation groups

Symmetric group S_n Show that S_3 is generated by $a = (12), b = (23)$ by drawing the Cayley digraph. How can you tell from your diagram that $S = \{a, b\}$ is a generating set?

Dihedral group D_n Prove that the dihedral group D_4 is nonabelian.

Cycle notation What is the product (i.e. composition) $(123)(345)$ in cycle notation?

1.2. Not on the quiz (but it would be useful for you to know)

Orbits (What is the difference between orbits and cycles?)

Cycles (But cycle notation will be used since it is shorter!)

Even, odd permutations

Alternating groups (But I need to finish the discussion of A_n on the last page of the notes.)

Semigroups, monoids

Annihilators

Equivalence relations.

1.3. Working though review problems.

1.3.1. Prove that the dihedral group D_4 is nonabelian.

Steps to do this problem:

(1) First write down the definition of D_4 and choose notation.
(2) What does it mean that the group is nonabelian?
(3) You need to be specific. Take two particular elements of the group using the notation you chose earlier and show by simple calculation that they do not commute.

1.4. Find other conditions on the determinant which will give subgroups of $GL(n, \mathbb{R})$.

(1) If H is a subgroup of $GL(n, \mathbb{R})$ and H contains an element with determinant 2 then H will also contain elements with determinant .....
(2) What set of determinants did you get?
(3) What can you say in general (without assuming (1) above) about the set $D = \{\det A \,|\, A \in H\}$?

1.4.1. Find a 2×2 matrix A of order 2 and determant -1. What group does A generate?

(1) First rewrite the problem. What does A look like? What does it mean that A has order 2?
(2) Make simplifying assumptions: insert zeros into the matrix.
(3) The second question is a little ambiguous. "The group generated by A" is not an answer. But it can be part of the answer. The notation $\langle A \rangle$ is also just notation.
(4) The answer should be: "The group generated by A, denoted $\langle A \rangle$ is the set ..... with operation given by .... "(Fill in the blanks)

1.4.2. Suppose that H, K are subgroups of an abelian group G. Then show that the set

$$HK = \{hk \,|\, h \in H, k \in K\}$$

is a subgroup of G.

(1) (identity) Show that $e \in HK$
(2) (inverse) Show that the inverse of hk is also in HK

(3) (closure) Take two elements of HK. (Choose notation carefully.) Show by a calculation that the product is in HK

1.4.3. The group $(\mathbb{Z}^2, +)$ of all points in the plane with integer coordinates is generated by two elements $(1,0)$ and $(0,1)$. Find one billion more pairs of generators.

(1) Since I asked for lots of pairs of generators, you need a formula. So, take two points and figure out what needs to happen for them to be generators.
(2) It helps to know that if $\langle a, b\rangle$ contains a generating set then it is equal to the entire group G. Why is that?
(3) If the first point is $(1,0)$ what can the second point be?

1.5. **Problems from the book.** p. 72 #3, 11,15.

p. 83 #35, 17,19 Note: the notation for the elements of D_4 in the book is terrible. Use: t^i instead of ρ_i and s, st, st^2, st^3 instead of $\mu_1, \delta_2, \mu_2, \delta_1$ respectively.

36: (rephrased) Find an example of a nonabelian group all of whose proper subgroups are abelian. Two of the groups mentioned above will work. But you have to explain why.