

1. MATH 30A, FALL 2009
Review for Quiz 1

Rules for the quiz: Closed book. 40 minutes. *You may bring notes written on a single piece of paper (both sides, letter size). You need to decide what should be written on your paper. You can't ask me about definitions. I am going to ask you!*

1.1. List of topics.

Binary operation Find an infinite number of binary operations on \mathbb{Z} .

There are lots of answers. For each number n you can define the operation which always gives n : $a * b = n$ for all $a, b \in \mathbb{Z}$. A more interesting answer would be: For each N take the operation: $a *_N b = a + b - N$. This is a group operation with identity N and inverse $a^{-1} = 2N - a$.

Isomorphism Find an isomorphism from $(\mathbb{Z}_2, +_2)$ to $GL(1, \mathbb{Z})$ There is only one isomorphism like this: Let $\phi(0) = I_1, \phi(1) = -I_1$. These multiply correctly since $1 = I_1$ is the identity of $GL(1, \mathbb{Z})$ and $(-I_1)^2 = I_1$.

Definition of group (Identity, inverse, associativity) Show that the set of all positive rational numbers a/b where a, b are positive odd integers is a multiplicative group. First we need to verify that multiplication defines a binary operation on this set. But this is clear since the product of odd numbers is odd. So $(a/b)(c/d) = ac/bd$ has odd numerator ac and odd denominator bd and all numbers are positive.

Next we need to verify that the operation is associative. However, we know that multiplication of rational numbers is associative.

Next we need the identity: $1/1$ is in the set and is clearly the identity.

Finally we need inverses: b/a is the inverse of a/b since $(b/a)(a/b) = ba/ab = 1/1$.

Abelian (commutative) groups Give an example of a noncommutative finite group. We did this in class. S_3 is a noncommutative finite group. It is finite since it has only 6 elements. It is not commutative since it contains the elements $a = (12)$ and $b = (123)$ which computation shows do not commute:

$$ab = (12)(123) = (23) \neq ba = (123)(12) = (13)$$

Subgroups Show that the set of rational number a/b where b is an odd integer is a subgroup of the group $(\mathbb{Q}, +)$.

(Choose notation.) Let $H = \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}$

We want to show that H is a subgroup of $(\mathbb{Q}, +)$. We will do this the long way.

- (1) The additive identity in \mathbb{Q} is 0 which is equal to $0/1$ and is therefore in H .
- (2) The inverse of a/b is $-a/b$ which is in H since b is still odd.
- (3) The sum of two elements in H is:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

The denominator bd is odd and, after dividing by the greatest common divisor of $ad + bc$ and bd we will still have an odd denominator since any divisor of an odd number is odd.

Since all three conditions are satisfied, H is an additive subgroup of \mathbb{Q} .

Determinant $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$ given by a condition on the determinant, namely $\det A = 1$. [Find other conditions on the determinant which will give subgroups of \$GL\(n, \mathbb{R}\)\$.](#)

We did this in class. Examples of conditions which give subgroups are $\det A > 0$, $\det A = 2^k$ for some $k \in \mathbb{Z}$ and the most complete answer is $\det A \in G$ where G is a subgroup of the multiplicative group of nonzero real numbers.

Cyclic groups \mathbb{Z}_n, U_n [Find all subgroups of \$\mathbb{Z}\$.](#)

The theorem is that all subgroups of cyclic groups are cyclic. Since \mathbb{Z} is cyclic, all subgroups have the form $\langle n \rangle = n\mathbb{Z}$. So, since $(-n)\mathbb{Z} = n\mathbb{Z}$, the subgroups of \mathbb{Z} are:

$$\{0\}, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, 5\mathbb{Z}, \text{etc.}$$

where $n\mathbb{Z}$ is the set of all multiples of n .

[Find all isomorphisms \$\mathbb{Z}_5 \rightarrow U_5\$. \(How many are there?\)](#)

The answer is that there are 4 isomorphisms. Let's call them $\phi_1, \phi_2, \phi_3, \phi_4$. The formula for ϕ_k is

$$\phi_k(1) = e^{2\pi k/5}$$

If $\phi : \mathbb{Z}_5 \rightarrow U_5$ is an isomorphism then $\phi(2) = \phi(1)\phi(1) = \phi(1)^2$, $\phi(3) = \phi(1)\phi(2) = \phi(1)^3$, $\phi(4) = \phi(2)\phi(2) = \phi(1)^4$ and $\phi(0) = 1$. So, ϕ is uniquely determined by $\phi(1)$. Since U_5 has only 5 elements, there are only 5 possibilities for $\phi(1)$. But one of them does not work, namely $\phi(1) = 1$ would not give an isomorphism since $\phi(0)$ must be also be equal to 1 and ϕ would not be 1-1 and thus not an isomorphism. The other 4 choices for $\phi(1)$ give isomorphisms.

Division algorithm ($k = qn + r$) [State the theorem.](#)

If n is a positive integer and k is any integer then there exist unique integers q and r so that $k = qn + r$ and $n > r \geq 0$.

Order of groups and elements. [Does a group of order \$n\$ always have an element of order \$n\$](#)

No, the Klein 4-group has order 4 but the elements have order 2 except the identity which has order 1.

Generators The group $(\mathbb{Z}^2, +)$ of all points in the plane with integer coordinates is generated by two elements $(1, 0)$ and $(0, 1)$. Find one billion more pairs of generators.

The pair of elements $(1, 0)$ and $(n, 1)$ generate \mathbb{Z}^2 for all integers n . To prove this take any $(a, b) \in \mathbb{Z}^2$. Then

$$(a, b) = (a - bn)(1, 0) + b(n, 1)$$

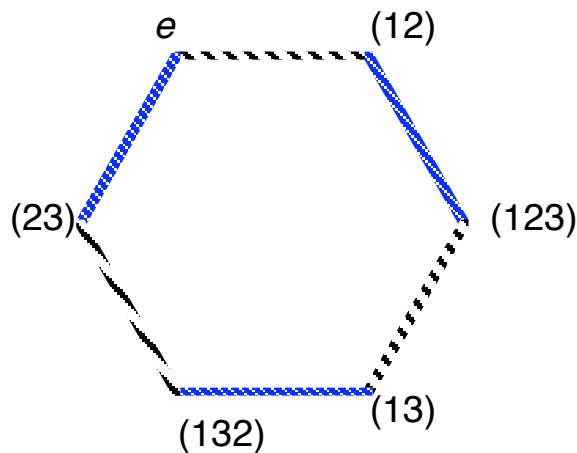
Cayley digraphs Draw the Cayley digraph for the group \mathbb{Z}_5 with generator 2.

This looks like a 5 pointed star.

Permutation groups

Symmetric group S_n Show that S_3 is generated by $a = (12), b = (23)$ by drawing the Cayley digraph. How can you tell from your diagram that $S = \{a, b\}$ is a generating set?

S is a generating set since the Cayley graph is connected. You can get from the identity e to any element of the group by a path. Along the path we multiply the group element by a generator or its inverse at each step. So, each element of the group is a product of a, b, a^{-1}, b^{-1} with repetitions.



Dihedral group D_n Prove that the dihedral group D_4 is nonabelian. We did that in class. Take any rotation a and reflection b and calculate ab, ba They will not be equal.

Cycle notation What is the product (i.e. composition) $(123)(345)$ in cycle notation?

$$(123)(345) = (12345)$$

1.2. Not on the quiz (but it would be useful for you to know)

Orbits (What is the difference between orbits and cycles?) Orbits are subsets of the set of letters being permuted. Cycles are permutations having at most one not-singleton orbit.

Cycles (But cycle notation will be used since it is shorter!)

Even, odd permutations

Alternating groups (But I need to finish the discussion of A_n on the last page of the notes.)

Semigroups, monoids

Annihilators

Equivalence relations.