2. MATH 30A, FALL 2009
Answers to Practice Quiz 2

2.1. Factor groups

(1) *What is the definition of a factor group?*
(2) *What are the elements of the factor group $GL(2,\mathbb{Z})/SL(2,\mathbb{Z})$? Explain how they are multiplied.*

Definition 2.1. Suppose that N is a normal subgroup of a group G. Then the **factor group** G/N is defined to be the set to all cosets gN of N in G with multiplication given by

$$(gN)(hN) = ghN$$

It is a theorem proved in class that this formula gives a well-defined binary operation on the set of cosets of N in G provided that N is normal in G. It is also an easy theorem that G/N is a group under this operation.

The factor group $GL(2,\mathbb{Z})/SL(2,\mathbb{Z})$ has only two elements. One is

$$E = SL(2,\mathbb{Z}) = \{g \in GL(2,\mathbb{Z}) \mid \det g = 1\}$$

and the other is the complement

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} SL(2,\mathbb{Z}) = \{g \in GL(2,\mathbb{Z}) \mid \det g = -1\}$$

Multiplication is given by $EE = E = TT$, $ET = TE = T$.

2.2. *Find all elements of order 2 in S_4.*

There are 9 elements of order 2. Six of them are the transpositions:

$$(12), (13), (14), (23), (24), (34)$$

and the other three permutations of order 2 are:

$$(12)(34), (13)(24), (14)(23)$$

Which ones are conjugate?

The first 6 are conjugate (to each other) and the last 3 are conjugate.

Use your answer to prove that

$$K := \{e, (12)(34), (13)(24), (14)(23)\}$$

is a normal subgroup of S_4.

Proof. This is the Klein four group V. It is a subgroup since every nontrivial element has order two and the product of any two of them is the third. For example:

$$(12)(34)\ (13)(24) = (14)(23)$$

K is normal since it contains all conjugates of all of its elements.

$$gKg^{-1} = \{e, (g(1)g(2))(g(3)g(4)), \text{etc}\} = K$$

□

2.3. The group S_3 acts on the set $\{1, 2, 3\}$

(1) *Find the stabilizer subgroup of the "letter" 3*

The stabilizer of 3 is the subgroup $H = \{e, (12)\}$ since these are the only permutations which do not move 3.

(2) *What is the orbit-stabilizer formula and how does it apply to this case?*

It says that the size of the orbit of $x \in X$ is equal to the index of the stabilizer of x. In this case, $x - 3$ with stablilizer $H = \langle(12)\rangle$ having index $6/2 = 3$. So there are 3 elements in the orbit:

$$S_3(3) = \{1, 2, 3\}$$

(Alternative question: *What is the orbit-coset correspondence in general and what is it in this case?*)

The orbit-coset correspondence says there is a 1-1 correspondence between elements gx of the orbit of $x \in X$ and left cosets gG_x of the stabilizer of x. The bijection ϕ is given by

$$\phi(gx) = gG_x.$$

In this particular case the correspondence is:

gx		gH
1	$\leftrightarrow$	$(13)H$
2	$\leftrightarrow$	$(23)H$
3	$\leftrightarrow$	H

2.4. Ring definitions

(1) What is the definition of a ring?
(2) What is the definition of a commutative ring with unity?
(3) Give an example of a ring which is not one of these (commutative ring with unity).

$M_2(\mathbb{Z})$, the ring of 2×2 matrices with coefficients in $\mathbb{Z}$ is a noncommutative ring with unity. $2\mathbb{Z}$ is a commutative ring without unity. $M_2(2\mathbb{Z})$ is a noncommutative ring without unity.

The ring $2\mathbb{R}$ has unity. Why?