1. Math 30a, Fall 2009
Worksheet 1

Examples

The set of Pythagorean triples can be made into a group in several ways. The easiest way is to allow negative numbers:

Definition 1.1. Define a primitive *Pythagorean triple* to be a triple of integers (a, b, c) where

$$a^2 + b^2 = c^2$$

a, b are relatively prime (do not have a common divisor) and c is positive. For example $(3, -4, 5)$ is a primitive Pythagorean triple according to this definition.

1.1. Problem: Use complex numbers to define a group operation on this set. What is the identity? What is the inverse?

1.2. Four bit addition is given by apply the rules $0 + 0 = 1, 0 + 1 = 1 + 0 = 1$ and $1 + 1 = 0$ to each column *without carrying*. For example:

$$\begin{array}{rr} & 0110 \\ + & 1011 \\ \hline & 1101 \end{array}$$

Prove that the set of sequences of four 0's and 1's forms a group under this operation. What is the identity? What is the inverse? (Ask me about 4-bit multiplication. Fun but very difficult to fully understand.)

1.3. Suppose that $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ is given by the equation $\phi(0) = 0$ and

$$\phi(x) = n - x$$

if $x \neq 0$. Is ϕ a bijection? an isomorphism?

Proofs

1.4. If A is a fixed invertible $n \times n$ real matrix then show that

$$\phi(X) := AXA^{-1}$$

gives an isomorphism $\phi : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$.

1.5. Give an example of a binary operation on a set with 2 elements which is associative but is not a group.

1.6. Suppose that a, b are elements of a group G and $a^2 = b^2$. Then, does it follow that $a = b$?