

1. INTRODUCTION

Algebra is the study of sets with binary operations. So, we will be talking about sets throughout the entire semester. I will spend the first two days reviewing set theory. Note that Math 23 is a prerequisite or corequisite for this course.

My main objective in this course is to teach students the language of Algebra: to understand the definitions and questions and be able to talk about it.

Review of set theory:

- a) Induction.
- b) Bijections and cardinality
- c) Power set
- d) Equivalence relations, congruence modulo n

1.1. **Induction.** Show by induction on n that

$$1 + 4 + 9 + 16 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The basis for the induction is: When $n = 1$ this statement is true.

Now suppose by induction that $n \geq 1$ and the statement holds for n . Then we need to show that it holds for $n + 1$. On the LHS we have:

$$\begin{aligned} 1 + 4 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{(n+1)(6n+6)}{6} \\ &= \frac{(n+1)[2n^2 + n + 6n + 6]}{6} \end{aligned}$$

On the RHS we have:

$$\begin{aligned} \frac{(n+1)(n+2)(2(n+1)+1)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)[2n^2 + 7n + 6]}{6} \end{aligned}$$

So, $LHS = RHS$ and the equation holds for $n + 1$.

1.2. **Bijection (a).** Prove that a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ whose derivative is always positive is 1-1 but not necessarily onto.

I used this example to explain the basic properties of mappings. First of all a *function* or *mapping* has three parts: a set A called the *domain* of f , a set B called the *codomain* of f and the function f . We write:

$$f : A \rightarrow B$$

For every $a \in A$ we get one element $f(a) \in B$. In set theory, where everything is a set, the function f is identified with its graph

$$G(f) = \{(a, b) \in A \times B \mid b = f(a)\}$$

which is explained below.

Problem (a) has two parts. The first part is to prove that any differentiable function with positive derivative is 1-1. The second part is to find an example of a function with these properties which is not onto.

I reviewed the definitions. *Surjective* or *onto* means that $f(x)$ gives all values on the right hand side of the arrow $f : \mathbb{R} \rightarrow \mathbb{R}$. For example, $f(x) = x^2$ is not onto \mathbb{R} . The formal definition is:

Definition 1.1. $f : A \rightarrow B$ is surjective or onto if, for every $b \in B$ there exists an $a \in A$ so that $f(a) = b$.

Someone came up with the example

$$f(x) = e^x$$

This is not surjective since e^x is always positive.

Definition 1.2. A function $f : A \rightarrow B$ is defined to be 1-1 if it sends two elements to two elements ("2-2" would be a better way to say this). In other words, any two distinct elements $a, b \in A$, $a \neq b$ go to two distinct elements of B : $f(a) \neq f(b)$.

So, to prove f is 1-1 we take two distinct elements of the domain $x_1, x_2 \in \mathbb{R}$. One of them will be bigger than the other, say $x_2 > x_1$. Then by the fundamental theorem of calculus we have

$$f(x_2) = f(x_1) + \int_{x_1}^{x_2} f'(x) dx$$

Since $f'(x) > 0$ for all x , its integral is positive:

$$\int_{x_1}^{x_2} f'(x) dx > 0$$

Therefore, $f(x_2) > f(x_1)$. In particular $f(x_2) \neq f(x_1)$. So f is 1-1.

1.3. Bijection (b). If there is a mapping of sets $f : A \rightarrow B$ which is 1-1 but not onto then what can you say about the cardinality of the sets A, B ?

If f is not onto then B has at least one more element than A . This means that

$$|A| < |B| \text{ if } A \text{ is a finite set.}$$

The correct answer in general is:

$$|A| \leq |B|$$

Here $|A|$ denotes the *cardinality* of the set A . This is the number of elements of A . However, it can be various degrees of infinity. Cardinality will not play a large role in this course. So, I don't want to prove theorems about cardinality. I just want to discuss the concept and history.

If A, B are infinite sets then it could happen that they have the same number of elements and there might be a 1-1 mapping which is not onto. In fact, this is the definition of an infinite set.

Definition 1.3. *A set A is infinite if there exists a mapping $f : A \rightarrow A$ which is 1-1 but not onto.*

For example, the infinite set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ has a 1-1 mapping $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ which is not onto given by $f(n) = 2n$.

1.4. Bijection (c). *Find a function $\mathbb{Z} \rightarrow \mathbb{Z}$ which is surjective but not 1-1.*

The first answer we got which I simplified here was:

$$f(n) = \begin{cases} n & \text{if } n < 2 \\ n - 1 & \text{if } n \geq 2 \end{cases}$$

Since $f(1) = 1$ and $f(2) = 1$, this function is not 1-1. Then I asked for a formula with only one equation which would be easier to type and someone gave the example:

$$f(n) = \left\lceil \frac{n}{2} \right\rceil$$

where $\lceil - \rceil$ means *round up* to the nearest integer.

1.5. Bijection (d). *Suppose that $f : A \rightarrow B$ and $g : B \rightarrow C$ are mappings of sets so that $g \circ f : A \rightarrow C$ is a bijection. Then what can you say about f and g ?*

The answer is: f is an injection and g is a surjection. I asked for a proof that g is surjective. Here is a diagram which may be helpful.

