1.6. **Axiom of Choice.** If $f : A \to B$ is surjective then for every $b \in B$ there is an element $a \in A$ so that $f(a) = b$. The *Axiom of Choice* says that we can put all of these together to form a function

$$a = s(b), \quad s : B \to A$$

Then the composition $f \circ s : B \to B$ is the identity:

$$(f \circ s)(b) = f(s(b)) = f(a) = b.$$

Surjective refers to each element $b \in B$ separately and find $a = s(b)$. The Axiom of Choice says that we can do it all at once (make an infinite number of random choice at one time).

1.7. **Power set.** If A is any set, the *power set* $\mathcal{P}(A)$ is the set of all subsets of A. The famous Cantor diagonalization argument shows that the cardinality of $\mathcal{P}(A)$ is always strictly greater than the cardinality of A. You don't need to know the proof of this statement.

The cardinality of the power set is $|\mathcal{P}(A)| = 2^{|A|}$. For example, a set with 3 elements has $2^3 = 8$ subsets.

1.8. **Relations and Cartesian products.** The *Cartesian product* $A \times B$ of two sets A, B is the set of all ordered pairs (a, b) where $a \in A, b \in B$. This is useful since the *graph* of a function $f : A \to B$ is a subset of the Cartesian product:

$$G(f) = \{(a, b) \in A \times B \,|\, b = f(a)\}$$

A *relation* $\mathcal{R}$ on a set A is defined to be any subset of the Cartesian product $A \times A$. If $(a, b) \in \mathcal{R}$ then we write $a\mathcal{R}b$. For example, the $\leq$ relation on $\mathbb{R}$ is the set of all pairs of real numbers (x, y) so that $x \leq y$.

If you shade in this set you note

1. The set includes the diagonal $\Delta = \{(x, x) \,|\, x \in \mathbb{R}\}$
2. The set is only above the diagonal and not below. So, it is not symmetric around the diagonal.

A relation $\mathcal{R}$ on a set A is called *reflexive* if $(a, a) \in \mathcal{R}$ for all $a \in A$. The relation $\leq$ is reflexive. A relation $\mathcal{R}$ on a set A is called *symmetric* if $(a, b) \in \mathcal{R}$ implies $(b, a) \in \mathcal{R}$. Or: $a\mathcal{R}b \Rightarrow b\mathcal{R}a$. Draw a picture of a symmetric relation on the set of real numbers. Explain why the drawing fits the definition.

Here is a problem which combines these concepts.

Show that $2^n > n^2$ for all integers $n \geq 5$. Give an interpretation in terms of sets. Does this make sense when n is infinite?

The proof is in the homework 1 instructions. The interpretation is:

If A is a set having at least 5 elements then the power set of A has strictly greater cardinality than the Cartesian product $A \times A$.

1.9. Partitions and equivalence.

Often sets are partitioned. For example, real numbers are often partitioned into positive, negative and zero:

$$\mathbb{R} = \mathbb{R}^+ \coprod \mathbb{R}^- \coprod \{0\}$$

Partitions have the property that every element of the set lies in exactly one "part" or "cell" and no cell is allowed to be empty.

One extremely important example that we will use is the partition of the set of integers according to their remainder after dividing by some number n. For example, if $n = 10$ then we can partition the set of positive integers into ten cells according to their last digit. All positive integers ending in 1 are in one cell, etc. So, the cells of this partition are:

$$\begin{array}{l} \{1, \ \ 11, \ \ 21, \ \ 31, \ \ , \cdots\} \\ \{2, \ \ 12, \ \ 22, \ \ 32, \ \ , \cdots\} \\ \{3, \ \ 13, \ \ 23, \ \ 33, \ \ , \cdots\} \\ \{4, \ \ 14, \ \ 24, \ \ 34, \ \ , \cdots\} \\ \qquad\qquad\qquad\qquad \text{etc.} \end{array}$$

Given a partition of a set A, we have a relation on this set which is "being in the same cell" So, elements of A are related if they are "cell-mates."

$$x \sim y \Leftrightarrow \ x \text{ and } y \text{ are in the same cell}$$

For example $x \sim y$ if x, y have the same last digit in base 10. The last digit is equal to the remainder when dividing by 10. We could take other bases and say $x \equiv_n y$ if x, y have the same remainder when divided by n. For example $x \equiv_2 y$ means that x, y have the same *parity* which means they are either both even or both odd.

Definition 1.4. *An* equivalence relation *of a set A is any relation which is reflexive, symmetric and transitive where* transitive *means $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.*

What does this mean geometrically on the graph?

Theorem 1.5. *A partition of a set gives an equivalence relation of being cell-mates and an equivalence relation on a set A gives a partition of the set into equivalence classes:*

$$[a] = \{x \in A \,|\, x\mathcal{R}a\}$$

The proof is on page 8 of the book. We will discuss the theory of equivalence relations again when we get to factor groups.