3. ISOMORPHIC BINARY STRUCTURES

Two binary structures $(S, *)$ and $(T, *)$ are called **isomorphic** if there is a bijection $\phi : S \to T$ so that

$$\phi(a * b) = \phi(a) * \phi(b)$$

for all $a, b \in S$. This equation is called the *homomorphism property.* Mathematicians also say "ϕ commutes with mutiplication." The notation is $(S, *) \cong (T, *)$.

The word "commutes" means $ab = ba$. In this case
$a =$ "ϕ of" and
$b =$ "the product of"
So:
(ϕ of)(the product of) a and b is equal to
(the product of)(ϕ of) a and b

A bijection with the homomorphism property is called an *isomorphism.*

Example 3.1. $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$. *The isomorphism $\phi : \mathbb{R} \to \mathbb{R}^+$ is given by $\phi(x) = e^x$. The homomorphism property is verified by the properties of exponents:*

$$\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\phi(y)$$

People say: "ϕ takes addition to multiplication."

3.1. showing that two structures are isomorphic.

To show that two binary structures are isomorphic:

$$(S, *) \cong (T, *)$$

we need to do four things:

(1) Write down a mapping $\phi : S \to T$.
(2) Show that ϕ is one-to-one.
(3) Show that ϕ is onto.
(4) Show that ϕ has the homomorphism property:

$$\phi(x * y) = \phi(x) * \phi(y)$$

for all $x, y \in S$.

The last step has a conceptual interpretation. Once we have a bijection $\phi : S \to T$, the binary operation on S will correspond to some binary operation on T, namely, given two elements of T, you take the corresponding elements of S, multiply in S, giving a product in S, and take the corresponding element of T.

Example 3.2. *Suppose that an isomorphism* $\phi : (\mathbb{R}, +) \to (\mathbb{R}, *)$ *is given by*

$$\phi(x) = 2x + 6$$

then what is the binary operation $*$ *?*

3.2. **structural properties.** To show that two binary structures are not isomorphic, we look at the *structural properties*. These are the properties shared by all isomorphic structures. For example, the cardinality of the set is a structural property. Commutativity and associativity are structural properties. Having an identity[1] is a structural property. The identity, if it exists is called e or e_S.

Example 3.3. *Take the binary structure* $(S, \cdot_8)$ *where* $S = \{1, 3, 5, 7\}$ *and* $\cdot_8$ *is multiplication modulo 8*

$\cdot_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

This set has the following structural properties:

(1) $|S| = 4$
(2) S has an identity $e = 1$
(3) The operation is commutative
(4) $x * x = e$ for all $x \in S$.

The last structural property shows: *This binary structure is not isomorphic to* $(\mathbb{Z}_4, +_4)$ since 0 is the identity for $+_4$ in $\mathbb{Z}_4$ and $1 +_4 1 = 2 \neq 0$.

3.3. **finding isomorphisms.** If you have binary structures with the same structural properties then you should suspect that they are isomorphic. To find an isomorphism, you need to match the structures.

Example 3.4. *Find an isomorphism* $\phi : (\mathbb{R}, \cdot) \cong (\mathbb{R}, *)$ *where* $*$ *is defined by*

$$x * y = xy + x + y$$

$(\mathbb{R}, \cdot)$ has identity 1 and annihilator 0. $(\mathbb{R}, *)$ has identity 0 and annihilator -1. So, ϕ if it exists must send 0 to -1 and 1 to 0.

[1] This is what I called "unity" last week. The identity is also called a "neutral element" and my word "unity" refers to a multiplicative identity.