7. Generating sets and Cayley digraphs

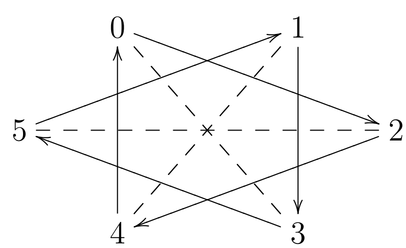
In a cyclic group G all elements are powers of a single element g which is called a *generator* of the group. Other groups such as the Klein four group V are not cyclic and require more than one generator.

Definition 7.1. *A set of elements of G is called a* generating set *and the elements are called* **generators** *if every element of G is a product of the generators and inverses of generators possibly with many repetitions. For example the elements of G look like: $b^5a^{-1}b^2a^{-9}b$ if a, b are the generators. The generators are said to "generate" the group. The product of no elements is by definition equal to the identity e.*

Examples:

(1) The Klein four group $V = \{e, a, b, c\}$ has generating set $\{a, b\}$. Also b, c generate V and a, c are generators of V.
(2) The set of all elements of a group G is a generating set.
(3) $2, 3$ generate $\mathbb{Z}_6$ but neither element by itself is a generator. Whether or not "g is a generator of G" depends on what the other elements in the set are.

Given a generating set we can draw a directed graph called the *Cayley digraph* with one vertex for each element of the group and one arrow for each generators starting at each vertex. So, for example $G = \mathbb{Z}_6$ with generator 1 is a hexagon. With generating set $\{2, 3\}$ it would look like this:



The solid arrows indicate addition by 2 and the dotted lines are addition by 3. Since 3 is its own inverse there is no arrow direction on those dotted line. For example $2 +_6 3 = 5$ and $5 +_6 3 = 2$. So adding 3 modulo 6 will send 2 and 5 to each other, whereas adding 2 sends $0 \to 2 \to 4 \to 0$ and $1 \to 3 \to 5 \to 1$.

The Cayley digraph is always connected. Why?

Question: In what way is the square the Cayley digraph for the Klein four group?

Question: Do 12 and 14 generate $\mathbb{Z}$? (The operation is addition.)

7.1. **more about Cayley digraphs.** The Cayley digraph is a directed graph with labels on the vertices and edges. A sequence of edges gives a **path** if the head (or target) of each edge is the tail (or source) of the next. The labels on the edges are composed *left to right*. Thus abc means a then b then c:

$$abc: \quad \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet$$

The labels on the vertices are group elements which change by multiplication on the right:

$$g \xrightarrow{a} ga \xrightarrow{b} gab \xrightarrow{c} gabc$$