9. Orbits, cycles and A_n

First, I will do the example to explain the concepts. Then we will go over the rigorous definitions. A permutation of n letters $\sigma \in S_n$ has "orbits" and "cycles". These are essentially the same thing except that orbits are sets and therefore *unordered* whereas cycles are "cyclically ordered". Here is an example similar to the one in the book. $\sigma \in S_8$ is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 7 & 4 & 1 & 5 & 3 \end{pmatrix}$$

This permutation has two cycles:

(1) $1 \to 8 \to 3 \to 6 \to 1$. This is a 4-cycle and is written: (1836).
(2) $2 \to 2$. This is a 1-cycle written: (2) and 1-cycles don't count!!
(3) $4 \to 7 \to 5 \to 4$. This is the 3-cycle (475).

9.1. orbits.

The orbits of σ are the three sets:

(1) $\mathcal{O}_1 = \{1, 3, 6, 8\}$
(2) $\mathcal{O}_2 = \{2\}$. This is a *singleton*.
(3) $\mathcal{O}_3 = \{4, 5, 7\}$

The orbits are disjoint sets whose union is the set being permuted, namely the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$:

$$A = \mathcal{O}_1 \coprod \mathcal{O}_2 \coprod \mathcal{O}_3$$

This is a *partition* of the set A. The parts are called *cells* of the partition and they are given by an equivalence relation "lying in the same orbit"

Definition 9.1. *If σ is a permutation of the set A and $a, b \in A$ are two elements, we say that a, b are* in the same orbit *if there exists an integer n so that $\sigma^n(a) = b$. This is an equivalence relation and the equivalence classes are called the* **orbits** *of σ. The* orbit *of any $a \in A$ is therefore the set:*

$$\{\sigma^n(a) \mid n \in \mathbb{Z}\}$$

In the example, $\sigma(1) = 8, \sigma^2(1) = 3, \sigma^3(1) = 6, \sigma^4(1) = \sigma^0(1) = 1$. So the orbit of 1 is the set $\{1, 8, 3, 6\} = \{1, 3, 6, 8\}$. This is also the orbit of 3.

Problem: Show that the orbits of a permutation σ are either disjoint or equal.

9.2. **cycles.** Going back to the example, the first orbit $\{1,3,6,8\}$ comes in the order 1836. We interpret this as a separate permutation where the other "letters" 2,4,5,7 are fixed:

$$(1836) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 4 & 5 & 1 & 7 & 3 \end{pmatrix}$$

Definition 9.2. *A* **cycle** *is defined to be a permutation having at most one orbit which is not a singleton. The* length *of a cycle is the number of elements in the largest orbit.*

The *support* of any permutation σ is defined to be the set of all letters x so that $\sigma(x) \neq x$. This is the set of letters *moved* by σ. The length of a cycle is the size of its support except if it is a 1-cycle.

Problem: How many 1-cycles does S_n have? How many 2-cycles does S_n have? How many 3-cycles does it have?

Definition 9.3. *2-cycles are called* **transpositions**.

Problem: Show that disjoint cycles commute. (Why is it obvious that they commute?)

Here is an example that shows that cycles can commute even if they are not disjoint:

$$\sigma = (12345), \quad \tau = (13524)$$

What does it mean that these commute? Why do you think they commute?

Theorem 9.4. *Every permutation of n letters can be written as a product*[2] *of disjoint cycles of length ≥ 2. These disjoint cycles are uniquely determined. But the product can be written in any order.*

Proof. Suppose that $\sigma \in S_n$. For each orbit $\mathcal{O}$ of σ which is not a singleton, the permutation σ permutes the elements of $\mathcal{O}$ in one cycle $\sigma_{\mathcal{O}}$:

$$\sigma_{\mathcal{O}}(x) = \begin{cases} \sigma(x) & \text{if } x \in \mathcal{O} \\ x & \text{otherwise} \end{cases}$$

Do this for every orbit of σ which is not a singleton. Then σ is a product of the cycles $\sigma_{\mathcal{O}_1}, \sigma_{\mathcal{O}_2}, \cdots, \sigma_{\mathcal{O}_k}$ since, for any letter x lies in exactly on orbit, say $\mathcal{O}_i$ and $y = \sigma(x)$ also lies in $\mathcal{O}_i$. Then $\sigma_{\mathcal{O}_j}(x) = x$ and $\sigma_{\mathcal{O}_i}(x) = y$. So, $\sigma_{\mathcal{O}_1}, \sigma_{\mathcal{O}_2}, \cdots, \sigma_{\mathcal{O}_k}$ sends x to y.

To show uniqueness, suppose that σ is written as a product of disjoint cycles. Then the supports of these cycles must be the orbits of σ which are not singletons. Then the cycles must be $\sigma_{\mathcal{O}}$ as above. □

[2]The product of no elements of a group is defined to be the identity e.