

9.3. more about cycles. Problems: Write the following permutation of 8 as a product of disjoint cycles.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 1 & 7 & 3 & 4 & 6 & 2 \end{pmatrix}$$

What is the order of an n -cycle? What is the order of σ ?

Lemma 9.5. *Every n -cycle can be written as a product of $n - 1$ transpositions³*

Proof. Here is the formula:

$$(a_1 a_2 a_3 \cdots a_n) = (a_1 a_2)(a_2 a_3)(a_3 a_4) \cdots (a_{n-1} a_n)$$

for example a 4-cycle can be written as a product of 3 transpositions:

$$(1357) = (13)(35)(57)$$

□

Since every permutation of n can be written as a product of disjoint cycles and every cycle is a product of transpositions, we get the following.

Theorem 9.6. *Every permutation of n can be written as a product of transpositions.*

How many transpositions do you need? Suppose that σ is a permutation of 100 with 4 orbits of size 10,20,30,40. Then, in the cycle decomposition, σ is a product of 4 cycles of size 10,20,30,40. These cycles can be written as a product of 9,19,29,39. So, σ can be written as a product of

$$9 + 19 + 29 + 39 = 96$$

transpositions. The formula is $100 - 4$ or $n - k$ where k is the number of orbits. What is k for the identity permutation?

9.4. sign of a permutation.

Definition 9.7. *The sign of a permutation σ of n letters is defined to be $(-1)^{n-k}$ where k is the number of orbits of σ . The permutation σ is defined to be odd if $n - k$ is odd (and the sign is -1) and σ is even if $n - k$ is even.*

Notice that σ can be written as a product of $n - k$ transpositions. So, an even permutation is a product of an even number of transpositions and an odd permutation is a product of an odd number of transpositions.

³Research problem: Show that there are exactly n^{n-2} different ways to write an n cycle as a product of $n - 1$ transpositions.

Theorem 9.8. *A permutation of n which is a product of m transpositions is even if m is even and odd if m is odd.*

Proof. Suppose that σ can be written as a product of m transpositions. Then we want to show that m has the same parity as $n - k$, i.e., that $n - k - m$ is always an even integer. We will prove this by induction on m .

Suppose that $m = 1$. Then σ is a 2-cycle. So it has one orbit of size 2 and the remaining $n - 2$ letters form the other orbits which are singletons. So, σ has $k = n - 1$ orbits and $n - k - m = n - (n - 1) - 1 = 0$ is even.

Now suppose the statement holds for m . Then we will prove it for $m + 1$. This means we assume that σ is a product of $m + 1$ transpositions:

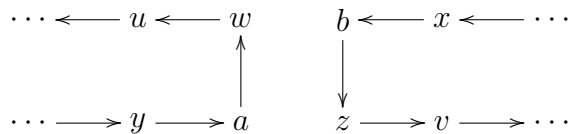
$$\sigma = \underbrace{\tau_1 \tau_2 \cdots \tau_m}_{\rho} \tau_{m+1} = \rho \tau_{m+1}$$

Since ρ is a product of m transpositions, we know by induction on m that $n - k - m$ is even where k is the number of orbits of ρ . To show that our statement holds for $m + 1$ it is enough to show that σ has either $k + 1$ or $k - 1$ orbits. Then

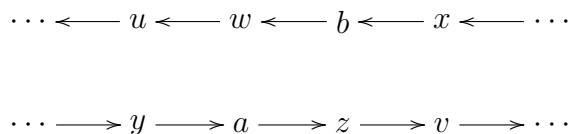
$$n - (\text{number of orbits of } \sigma) - (m + 1) = n - (k \pm 1) - (m - 1)$$

will be even.

The transposition τ_{m+1} is a 2-cycle, say $\tau_{m+1} = (ab)$. There are two cases: either a, b lie in different orbits of ρ or they lie in the same orbit of ρ . In the first case, the two orbits will “fuse” and become one orbit. So $\rho\tau_{m+1}$ will have $k - 1$ orbits. In the second case the orbit containing a, b will split into two orbits making $k + 1$ orbits for $\rho\tau_{m+1}$. Here is a diagram which illustrates both cases.



becomes:



□

9.5. the alternating group A_n .

Definition 9.9. *The alternating group on n letters, denoted A_n , is defined to be the subgroup of S_n consisting of all even permutations of n .*

To show that this is a subgroup:

- (1) A_n contains the identity since the identity has $k = n$ orbits.
- (2) A_n is closed under composition: If σ, τ are even then they can be written as a product of even numbers of transpositions, say $2s$ and $2t$. Multiplying them we get $\sigma\tau$ written as a product of $2s + 2t$ transpositions. So, it is even.
- (3) If σ is a product of $2s$ transpositions $\sigma = t_1 t_2 \cdots t_{2s}$ then the inverse is the product of the same $2s$ transpositions written backwards:

$$\sigma^{-1} = t_{2s} t_{2s-1} \cdots t_2 t_1$$

Why is that?

Problem: Show that the order of A_n is $n!/2$, i.e., A_n has exactly half the elements of S_n . Do the other elements of S_n form another subgroup?

Show that (for $n \geq 2$) S_n is the disjoint union of A_n and the set

$$(12)A_n = \{(12)\sigma \mid \sigma \in A_n\}$$

The alternating group A_4 contains a group isomorphic to the Klein 4-group V . It is

$$K = \{e, (12)(34), (13)(24), (14)(23)\}$$

Problem: show that any product of 3-cycles is even.