13. HOMOMORPHISMS

Definition 13.1. *A* homomorphism $\phi : G \to H$ *is a mapping between groups satisfying the condition*

$$\phi(gh) = \phi(g)\phi(h)$$

for all $g, h \in G$.

We will look at examples and study the elementary properties (in terms of elements) and the structural properties in terms of subgroups. The difference between homomorphisms and isomorphisms is that homomorphisms need not be surjective nor do they have to be injective (1-1) and this leads to some very interesting properties which lie at the heart of the structural theory of groups.

Example 13.2. *Examples of homomorphisms:*

1. $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ *where* $\mathbb{R}^\times = \{x \in \mathbb{R} \mid x \neq 0\}$ *is the multiplicative group of nonzero real numbers. This homomorphism is surjective but not injective.*
2. *If* G *is any group and* $g \in G$ *then the mapping* $\phi : \mathbb{Z} \to G$ *given by* $\phi(n) = g^n$ *is a homomorphism which is neither surjective nor injective.*
3. $sgn : S_n \to \{1, -1\}$. *This example defines the alternating group:* $A_n := \{\sigma \in S_n \mid sgn(\sigma) = 1\}$.

Basic properties of homomorphisms:

1. $\phi(e_G) = e_H$. Proof: Take $g = h = e$ then
$$\phi(e)e_H = \phi(e) = \phi(ee) = \phi(e)\phi(e)$$
By cancellation we get $\phi(e) = e$.
2. $\phi(g^{-1}) = \phi(g)^{-1}$. Proof:
$$e = \phi(e) = \phi(gg^{-1}) = \phi(g)\phi(g^{-1}).$$
3. $\phi(g^n) = \phi(g)^n$. Pf: By induction on n.

Structural properties of homomorphisms depend on the following definitions and theorems.

Definition 13.3. *A subgroup* $N \leq G$ *is called* normal *if*

$$gNg^{-1} = N$$

for all $g \in G$. *The notation is* $N \trianglelefteq G$.

The equation $gNg^{-1} = N$ is the same as the equation

$$gN = Ng$$

In other words, left cosets are the same as right cosets. From this interpretation we know from the example that we had earlier that $H = \{e, (12)\}$ is not a normal subgroup of S_3 since the left cosets are different from right cosets. Let's examine the definition. Put $g = (13)$. Then $g^{-1} = (13)$ so

$$gHg^{-1} = (13)H(13) = \{(13)(13), (13)(12)(13)\} = \{e, (23)\}$$

This is not equal to $H = \{e, (12)\}$. So, H is not normal.

Normal subgroups are related to homomorphisms in the following way. This is a definition and a theorem.

Theorem 13.4. *If $\phi : G \to H$ is a homomorphism then the set of all $g \in G$ which go to e in H is a normal subgroup. It is called the* **kernel** *of ϕ.*

$$\ker \phi := \{g \in G \mid \phi(g) = e\} \trianglelefteq G.$$

Proof. The kernel of ϕ is a subgroup of G since

(1) it contains e since $\phi(e) = e$
(2) If $g \in \ker \phi$ then $\phi(g^{-1}) = \phi(g)^{-1} = e^{-1} = e$. So, $g^{-1} \in \ker \phi$.
(3) If $g, h \in \ker \phi$ then $\phi(gh) = \phi(g)\phi(h) = ee = e$. So, $gh \in \ker \phi$.

To prove normality suppose that $h \in \ker \phi$. Then $\phi(h) = e$ so

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} = \phi(g)e\phi(g)^{-1} = e$$

so $ghg^{-1} \in \ker \phi$ for all $g \in G$. So $\ker \phi$ is normal in G. □

(1) The kernel of $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ is

$$\ker \det = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\} = SL(n, \mathbb{R})$$

(3) The kernel of $sgn : S_n \to \{1, -1\}$ is A_n.
(2) What is the kernel of $\phi : \mathbb{Z} \to G$ given by $\phi(n) = g^n$?

Theorem 13.5. *The image of any homomorphism $\phi : G \to H$ is a subgroup of H.*

Proof. The image $\operatorname{im} \phi$ is the set of all $\phi(g)$ where $g \in G$.

(1) $\operatorname{im} \phi$ contains $\phi(e) = e$.
(2) Any element of $\operatorname{im} \phi$ has the form $\phi(g)$. So, $\phi(g)^{-1} = \phi(g^{-1}) \in \operatorname{im} \phi$.
(3) The product of any two elements of $\operatorname{im} \phi$ is: $\phi(g)\phi(h) = \phi(gh) \in \operatorname{im} \phi$.

So, $\operatorname{im} \phi$ is a subgroup of H. □

The image of $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ is $\mathbb{R}^\times$ since any nonzero real number x is the determinant of the diagonal matrix with diagonal entries $x, 1, 1, \cdots, 1$. What are the images in the other two examples?