

One point which I forgot to make.

Lemma 13.6. *A subgroup N of G is normal if*

$$gNg^{-1} \subseteq N$$

for all $g \in G$.

This was used in the proof of Theorem 13.4. (We started with an element $h \in \ker \phi$ and showed that $ghg^{-1} \in \ker \phi$. This only proves that $gKg^{-1} \subseteq K$ for $K = \ker \phi$.

Proof. If $gNg^{-1} \subseteq N$ for all g then, in particular, this equation is true for g^{-1} . So

$$(g^{-1})N(g^{-1})^{-1} = g^{-1}Ng \subseteq N$$

Multiply by g on the left and g^{-1} on the right to get:

$$N \subseteq gNg^{-1}.$$

So, the two sets N, gNg^{-1} are contained in each other. So they are equal. \square

Problems

- (1) Use Lagrange to show that any group of prime order is cyclic.
- (2) If $|G| = pq$ where p, q are prime then any proper subgroup is cyclic.
- (3) If $o(g) = 2$ and g is the only element of G of order 2, then $\langle g \rangle$ is a normal subgroup of G .
- (4) If G is a finite group of order $2n$ then any subgroup of order n is normal.

Conjugation In class we talked about what happens when H is a subgroup of G which is not normal. Then gHg^{-1} is a subgroup of G which is different from H for some g . (It could be the same, for example $eHe^{-1} = H$.)

The groups H and gHg^{-1} are isomorphic. The isomorphism

$$\phi_g : H \rightarrow gHg^{-1}$$

is given by $\phi_g(h) = ghg^{-1}$. This is a homomorphism since

$$\phi_g(h)\phi_g(k) = ghg^{-1}gkg^{-1} = ghkg^{-1} = \phi_g(hk)$$

and it has an inverse given by $\phi_{g^{-1}}$:

$$\phi_{g^{-1}}\phi_g(h) = \phi_{g^{-1}}(ghg^{-1}) = g^{-1}ghg^{-1}(g^{-1})^{-1} = h$$

The isomorphism ϕ_g is called **conjugation by g** and gHg^{-1} is called a **conjugate** of H . So, N is normal if N is “invariant under conjugation.”