15. Simple groups

If G is a group then the trivial group $\{e\}$ and the whole group G are normal subgroups of G. The group G is defined to be **simple** if there are no other normal subgroups and $G \neq \{e\}$. (You could rephrase this to say G is simple if it has exactly two normal subgroups.)

Theorem 15.1. *The cyclic group $\mathbb{Z}_n$ is prime if and only if n is prime.*

Proof. Suppose $n = p$ is prime and H is a subgroup of $\mathbb{Z}_p$ then $m = |H|$ divides p. So $m = p$ or $m = 1$. So, $H = G$ or $H = \{0\}$. So, $\mathbb{Z}_p$ is simple.

Conversely, suppose that $G = \mathbb{Z}_n$ where n is not prime, say $n = pq$ where $1 < p < n$. Then

$$H = p\mathbb{Z}_n = \{0, p, 2p, \cdots, (q-1)p\}$$

is a proper normal subgroup of G. So, $\mathbb{Z}_n$ is not simple. □

Problem: Suppose that n is a positive integer. What is $\mathbb{R}/n\mathbb{R}$?

Problem: Does this mean that $(\mathbb{R}, +)$ is a simple group?

Problem: Compute: $\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle(2,3)\rangle$. Here $\langle(2,3)\rangle$ is the cyclic subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_6$ generated by the element $(2,3)$. [Hint: Consider the homomorphism $\phi : \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_{12}$ given by $\phi(a,b) = 3a +_{12} 2b$. What are the kernel and image of ϕ?]

Theorem 15.2. *The alternating group A_n is simple for all $n \geq 5$.*

I won't explain the proof of this (it is in the book). I want to explain instead why we care about simple groups.

The concept is that the simple groups are the "building blocks" out of which all finite groups can be constructed. If a group G is not simple, then G has a normal subgroup N with factor group G/N and we can reconstruct G out of these two pieces: $N, G/N$.

Definition 15.3. *We say that G is an* **extension** *of N by Q if N is a normal subgroup of G and Q is isomorphic to the quotient group G/N.*

Analogy: Simple groups are the "atoms" and all other finite group are "molecules" built out of these atoms.

15.1. Center of a group.

If G is a nonabelian group then it has elements a, b which do not commute: $ab \neq ba$. But, it may have an element c which commutes with everything: $cg = gc$ for all $g \in G$. (Can you think of such an element?) Such an element is called *central.*

Definition 15.4. *The* **center** *$Z(G)$ of G is defined to be the set of all $c \in G$ which commutes with every element of G:*

$$Z(G) = \{c \in G \mid cg = gc \ \forall g \in G\}$$

Theorem 15.5. *The center of G is a normal subgroup of G.*

Example 15.6. *For example, the center of $GL(2,\mathbb{R})$ is given by:*

$$Z(GL(2,\mathbb{R})) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{R}^\times \right\}$$

Example 15.7. *The center of D_4 is the subgroup generated by t^2, the $180°$ rotation. Since D_4 has only 8 elements you can go through the list and see that only e and t^2 are central.*

Proving that $Z(G)$ is a normal subgroup is very straightforward. You know how to show it is a subgroup. How do you show it is normal?

Problem: Show that the center of S_n is trivial for $n \geq 3$. [Hint: Suppose that $\sigma \in S_n$ is central and nontrivial. Then σ moves at least one letter, say $\sigma(x) = y$ where $x \neq y$. Now find another permutation τ so that $\sigma\tau \neq \tau\sigma$.]

15.2. **Commutator subgroup.** If $a, b \in G$ then the **commutator** of a and b is given by:

$$[a, b] := aba^{-1}b^{-1}$$

Problem: Show that a, b commute if and only if their commutator is trivial (equal to the identity e).

Two key points about commutators are

(1) If the normal subgroup N contains all the commutators of G then G/N is abelian: $aba^{-1}b^{-1} \in N$ means

$$N = aba^{-1}b^{-1}N$$

Mutiply both of these on the right by baN to get

$$(bN)(aN) = baN = aba^{-1}b^{-1}NbaN = abN = (aN)(bN)$$

(2) Conjugates of commutators are commutators:

$$g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$$

Definition 15.8. *The* **commutator subgroup** *G' of G is defined to be the subgroup of G generated by the commutators. This is a normal subgroup of G with abelian quotient G/G'.*

Example 15.9. *A_n is the commutator subgroup of S_n for all $n \geq 1$.*

Example 15.10. *$SL(n,\mathbb{R})$ is the commutator subgroup of $GL(n,\mathbb{R})$ for all $n \geq 2$.*

Theorem 15.11. *If N is a normal subgroup of G then G/N is abelian if and only if N contains G'.*