16. Group actions on sets

(Lecture by Bong Lian, notes by Maxim Starobinets.)

We try to analyze a group G by representing it as a symmetry of a set (in particular the dihedral group D_n and symmetric group S_n). That is, elements of G are represented as bijections of a particular set X (to itself: $X \to X$) in such a way that the group *multiplication* corresponds to composition of bijections and group *inversion* corresponds to taking the inverses of bijections.

16.1. Definition.

Definition 16.1. *If G is a group and X is a set, a G-***action** *on X (also called an* action of G on the set X*) is a mapping*

$$* : G \times X \to X \quad (g, x) \mapsto gx$$

so that

1. *$ex = x$ for all $x \in X$*
2. *$(g_1 g_2)x = g_1(g_2 x) \quad \forall x \in X, \forall g_1, g_2 \in G$.*

*In that case we say that X is a G-***set**

Example 16.2. *Take any set X and recall that S_X is the set of all bijections of X to itself. So, $\sigma \in S_X$ iff σ is a map $\sigma : X \to X$, which is 1-1 and onto. S_X is a group under composition and S_X acts on X by $S_X \times X \to X$, $(\sigma, x) \mapsto \sigma(x)$. Note that*

1. *$(e, x) \mapsto e(x) = x$ since the identity element of S_X is the identity mapping on X.*
2. *$(\sigma\tau)(x) = (\sigma \circ \tau)(x) = \sigma(\tau(x))$ since multiplication in S_X is given by composition of maps.*

Example 16.3. *Choose $X = \mathbb{Z}_n$. Then $S_X = S_n$ is the group of permutation of n letters.*

Let X be any G-set and $H \leq G$. Then X is also an H-set. So:

Any subgroup $H \leq S_X$ acts on the set X.

Definition: If X is a G-set then for all $g \in G$ define

$$\sigma_g : X \to X \quad x \mapsto gx \quad \forall x \in X$$

In other words, $\sigma_g(x) = gx$.

Claim: σ_g is a bijection ($\sigma_g \in S_X$).

Proof. $\sigma_{g^{-1}}$ is the inverse of σ_g:

$$(\sigma_{g^{-1}} \circ \sigma_g)(x) = g^{-1}gx = x \quad \text{and} \quad (\sigma_g \circ \sigma_{g^{-1}})(x) = x$$

So, σ_g is a bijection. □

So, from any G-set X we get a mapping

$$\phi : G \to S_X, \quad g \mapsto \sigma_g$$

Are the following equal?

$$\phi(g_1)\phi(g_2) = \sigma_{g_1} \circ \sigma_{g_2} \qquad \phi(g_1 g_1) = \sigma_{g_1 g_2}$$

Take any $x \in X$. Then

$$\sigma_{g_1 g_2}(x) = (g_1 g_2)(x) = g_1(g_2(x)) = g_1 \sigma_{g_2}(x) = \sigma_{g_1}(\sigma_{g_2}(x)) = (\sigma_{g_1} \circ \sigma_{g_2})(x)$$

So, ϕ is a homomorphism. This proves the following theorem.

Theorem 16.4. *Let X be a G-set. For $g \in G$, the map $\sigma_g : X \to X$, $x \mapsto gx$ is a bijection of X to itself. The map $\phi : G \to S_X$, $g \mapsto \sigma_g$ is a group homomorphism.*

Question: Can we recover the G-action on X from the homomorphism $\phi : G \to S_X$? Does ϕ always come from a G-action on X?

Yes: The corresponding action is given by

$$* : G \times X \to X, \quad (g, x) \mapsto gx := \phi(g)(x)$$

(1) $ex =_? x \ \forall x \in X$

$$ex := \phi(e)(x) = id_X(x) = x$$

($\phi(e) = e = id$ in S_X because ϕ is a homomorphism.)

(2) $(g_1 g_2)(x) =_? g_1(g_2(x))$:

$$\begin{aligned}(g_1 g_2)(x) &= \phi(g_1 g_2)(x) = [\phi(g_1)\phi(g_2)](x) \\ &= \phi(g_1)[\phi(g_2)(x)] = \phi(g_1)(g_2 x) = g_1(g_2(x))\end{aligned}$$

So, we get an action $*$ which is completely specified by the homomorphism ϕ. By the theorem, an action gives a homomorphism. So, specifying a G-action on X is equivalent to specifying a group homomorphism $\phi : G \to S_X$.

16.2. **Substructures of G-actions on X.** Let X be a G-set and let $\phi : G \to S_X$ be the corresponding group homomorphism. Then

$$\{g \in G \,|\, gx = x \ \forall x \in X\} \leq G$$

But, $gx = x \iff \phi(g)(x) = x$. So, $\phi(g) = id_X$ (identity bijection). So, the set above is

$$\{g \in G \,|\, \phi(g) = id_X\} = \ker \phi$$

is a normal subgroup of G. By the Isomorphism Theorem 14.3, ϕ induces a new group homomorphism

$$\overline{\phi} : G/\ker(\phi) \to S_X, \quad g\ker(\phi) \mapsto \phi(g)$$

(Wednesday, first review:)
X is a G-set
$\phi : G \to S_X$ is the associated group homomorphism
$\ker \phi \trianglelefteq G$ is a normal subgroup
The induced map

$$\mu = \overline{\phi} : G/\ker \phi \to S_X$$

makes X a $(G/\ker \phi)$-set.

Definition 16.5. (1) *A G-action on X is* **faithful** *if* $\ker(\phi) = \{e\}$.
(2) *A G-action on X is* **transitive** *if* $\forall x_1, x_2 \in X \ \exists g \in G$ *so that* $gx_1 = x_2$.

Fact: G acts transitively on X iff $\phi(G)$ acts transitively on X.

Proof. Suppose G acts transitively on X. Take $x_1, x_2 \in X$. By supposition, $\exists g \in G$ s.t. $gx_1 = x_2$. But $gx_1 = \phi(g)x_1$, so $\exists \phi(g)$ s.t. $\phi(g)x_1 = x_2$. The other direction is analogous. □

Example 16.6. *The dihedral group $D_4 = \{e, t, t^2, t^3, s, st, st^2, st^3\}$ acts on $X = \{1, 2, 3, 4\}$ (the set of vertices of a square). D_4 also acts on the set of edges $E = \{s_1, s_2, s_3, s_4\}$ on the set of meridians $M = \{m_1, m_2\}$ diagonals $D = \{d_1, d_2\}$ and the center point C by:*

	1	2	3	4	s_1	s_2	s_3	s_4	m_1	m_2	d_1	d_2	C
σ_t	2	3	4	1	s_4	s_1	s_2	s_3	m_2	m_1	d_2	d_1	C
σ_s	2	1	4	3	s_1	s_4	s_3	s_2	m_1	m_2	d_2	d_1	C
σ_{st}	1	4	3	2	s_2	s_1	s_4	s_3	m_2	m_1	d_1	d_2	C

(See Figure 16.9 on page 156.) The action of D_4 on X and E are faithful and transitive. The action of D_4 on M, D, C are not faithful (t^2 is in the kernel of all 3 of these actions) but they are transitive actions.

16.2.1. *isotropy subgroup.*

Definition 16.7. *Let X be a G-set and $g \in G$. Define:*

$$X_g := \{x \in X \mid gx = x\} \subseteq X$$

*This is the g-**fixed subset** of X. For $x \in X$ define:*

$$G_x := \{g \in G \mid gx = x\}$$

This is the **stabilizer** *or* isotropy subgroup *of x in G.*

Theorem 16.8. *If X is a G-set, given $x \in X$, $G_x \leq G$.*

Proof. Let $g_1, g_2, g \in G_x$ Then

(1) $e \in G_x$ since $ex = x$
(2) $g_1 g_2 \in G_x$ since $g_1 g_2(x) = g_1 x = x$
(3) $g^{-1} \in G_x$? we know that $gx = x$ and we want to know[6] that $g^{-1}x = x$. So, take the equation we know and act on both sides by g^{-1}:
$$g^{-1}(gx) = x = g^{-1}x.$$

□

16.2.2. *orbits.*

Definition 16.9. *Let X be a G-set. A G-**orbit** in X is a subset of X given by*

$$Gx = \{gx \mid g \in G\}$$

for some $x \in X$.

The same orbit can be written in different ways. For $x, y \in X$ when is $Gx = Gy$?

Lemma 16.10. *$Gx = Gy$ iff $y \in Gx$*

Proof. Suppose that $y \in Gx$. Then $y = gx$ for some $g \in G$. For any $g'y \in Gy$ we have

$$\underbrace{g'y}_{\in Gy} = \underbrace{g'g}_{\in G}\, x$$

So $Gy \subseteq Gx$. Also, $y = gx \Rightarrow g^{-1}y = x$. So, $x \in Gy$ which implies $Gx \subseteq Gy$. So, $Gx = Gy$.

Conversely suppose that $Gx = Gy$. Then $y \in Gy$ since $y = ey$. So, $y \in Gx$. □

[6] Many times we explain things by starting with what we don't know and ending up with something we know. That is not a proof. You need to rewrite the equation backwards starting with what we know.

Theorem 16.11 (Orbit-cosets correspondence)**.** *Let X be a G-set and $x \in X$. Then there is a bijection:*

$$Gx \xrightarrow{\cong} G/G_x$$

from the G-orbit of x to the set of all left cosets of G_x. In particular, if G is finite then $|Gx|$ divides G.

Proof. The second assertion follows from the first assertion and Lagrange's theorem.

The first assertion: the bijection is given by $gx \mapsto gG_x$. This is well defined since (follow arrows to the right)

$$gx = hx \iff h^{-1}gx = x \iff h^{-1}g \in G_x \iff gG_x = hG_x$$

This also shows the mapping is 1-1 and onto. (Follow arrows to the left to show 1-1. Onto is obvious.) □

Theorem 16.12 (Partition property of orbits)**.** *Let X be a G-set. Then any two distinct G-orbits in the G-set are disjoint.*

Proof. The *contrapositive* of the theorem is: $Gx \cap Gy \neq \emptyset \Rightarrow Gx = Gy$. So, suppose $Gx \cap Gy$ is nonempty and $z \in G_x \cap G_y$. Then $z \in Gx$ and $z \in Gy$. So, by the lemma, $Gz = Gx$ and $Gz = Gy$. So, $Gx = Gy$. □

So the different orbits of X are disjoint and give a partition of X:

$$X = \coprod Gx_i$$

(Disjoint union of different orbits Gx_i.)

Example 16.13. $D_4 = \{e, t, t^2, t^3, s, st, st^2.st^3\}$, $X = \{1, 2, 3, 4\}$ *Take $x = 1 \in X$. The stabilizer of 1 is*

$$G_1 = \{e, st\}$$

The orbit is $G1 = X$ with $|G1| = 4 = 8/2 = |G : G_1|$ elements since $G1$ has the same cardinality as the set of left cosets of G_1 in G.