For example, $x = (12) \in S_3$ has 3 conjugates $(12), (23), (13)$ and its centralizer $C((12)) = \{e, (12)\}$ has 2 elements and these two numbers give $2 \cdot 3 = 6 = |S_3|$.

Putting these together we get:

$$|G| = \sum_{i=1}^{c} |G \cdot x_i| = \sum_{i=1}^{c} |G : C(x_i)|$$

In the case of $G = S_3$ this is:

$$\begin{array}{ccccccc} |S_3| & = & |S_3 : C(e)| & + & |S_3 : C((12))| & + & |S_3 : C((123))| \\ 6 & = & 1 & + & 3 & + & 2 \end{array}$$

The first number is 1 since $C(e) = S_3$.

Problem: a) Show that $C(g) = G$ if and only if $g \in Z(G)$.

b) Show that every element of $Z(G)$ is in its own conjugacy class.

This implies that the number of times the index $|G : C(x_i)|$ is equal to 1 is the number of elements in the center $Z(G)$. Call this $z = |Z(G)|$. For example, $z = |G|$ if G is abelian.

Theorem 17.3 (class formula)**.** *If G is a finite group then*

$$|G| = |Z(G)| + \sum \underbrace{|G : C(x_i)|}_{\neq 1}$$

Corollary 17.4. *If the order of G is a power of a prime: $|G| = p^k, k \geq 1$ then G has a nontrivial center.*

Proof. In the class formula, all the numbers divide p^k. So, the numbers which are not 1, such as $|G|$ and $|G : C(x_i)|$ are multiples of p. Therefore, $|Z(G)|$ is divisible by p. But $e \in Z(G)$. So $Z(G)$ must have at least p elements. □

For example D_4 has $8 = 2^3$ elements and its center has $p = 2$ elements.

Corollary 17.5. *There are no nonabelian simple p-groups (groups whose order is a power of the prime p).*

Proof. Any nontrivial p-group P has a nontrivial center $Z(P)$ which we know is a normal subgroup of P. If P is simple then we must have $P = Z(P)$. So, P is abelian (which implies P has order p. Why?) □

17.2. **Burnside's theorem.** This is a formula for counting the number of orbits of a finite group acting on a finite set. (From Maxim's notes on Bong Lian's lecture.)

Theorem 17.6 (Burnside)**.** *If G is a finite group and X is a finite G-set then*

$$(\#G\text{-orbits in } X) \cdot |G| = \sum_{g \in G} |X_g|$$

Proof. We count the number of elements in the same set S in two different ways.

$$S = \{(g,x) \in G \times X \mid gx = x\}$$

(1) "Freeze" g. For $g_1 \in G$ put $S_{g_1} := \{(g_1, x) \in S\} \subseteq S$. Then

$$S = \coprod_{g \in G} S_g$$

since $S_{g_1} \cap S_{g_2} = \emptyset$ if $g_1 \neq g_2$. For each $g \in G$ there is a bijection:

$$S_g = \{(g,x) \in S\} \xleftrightarrow{1:1} X_g$$

given by

$$(g,x) \xleftrightarrow{1:1} x$$

So, the size of the set S is given by

$$|S| = \sum_{g \in G} |S_g| = \sum_{g \in G} |X_g|$$

(2) "Freeze" x. For $x \in X$ put

$$S^x := \{(g,x) \in S\} = \{(g,x) \mid gx = x\}$$

Then S is the disjoint union of S^x and we have a bijection

$$S^x \xleftrightarrow{1:1} G_x \qquad (g,x) \xleftrightarrow{1:1} g$$

So, the size of S is also given by

$$|S| = \sum_{x \in X} |S^x| = \sum_{x \in X} |G_x|$$

(3) $|G|/|G_x| = |Gx|$ (orbit-coset correspondence). Solve for $|G_x|$ to get:

$$|G_x| = \frac{|G|}{|Gx|}$$

For any x we get an orbit Gx. But we get the same orbit for each $y \in Gx$. If r is the number of orbits and we name the orbits $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_r$. Then $X = \coprod \mathcal{O}_i$ and

$$\sum_{x \in X} \frac{1}{|Gx|} = \sum_{i=1}^{r} \underbrace{\sum_{x \in \mathcal{O}_i} \frac{1}{|Gx|}}_{=1 \text{ since } |Gx| = |\mathcal{O}_i|} = r$$

Putting (2), (3) together we get

$$|S| = \sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|Gx|} = |G| \cdot \sum_{x \in X} \frac{1}{|Gx|} = |G| \cdot r$$

Compare with (1): $|S| = \sum_{g \in G} |X_g|$ to get

$$|G| \cdot (\# \text{ of orbits}) = \sum_{g \in G} |X_g|$$

□

Example 17.7. *How many different dice (6-face) can one make? (Consider any rotation of a die as the same.) The group of rotations of the cube has 24 elements. They form a subgroup of S_8 (since G acts faithfully on the set of vertices) and G forms a subgroup of S_6 since it acts faithfully on the set of faces.*

Let X be the set of all possible ways of marking the cube (with $1, 2, \cdots 6$ spots on the 6 sides) including rotationally equivalent markings. Then $|X| = 6! = 720$. The number of distinguishable dice is

$$\# \textit{ of orbits} = \frac{1}{24} \sum_{g \in G} |X_g|$$

$$= \frac{1}{24} |X_e| = \frac{1}{24} \cdot 720 = 30$$

since X_g is empty for $g \neq e$.

Example 17.8. *(From Homework 8 handout) Each of the 8 corners of a cube is to be tipped with one of four colors, each of which may be used on any number of corners. Find the number of distinguishable markings. Use the hint: the group of rotations of the cube has 24 elements consisting of the identity, 9 which leave a pair of opposite faces invariant, 8 which leave a pair of opposite vertices invariant and 6 leaving a pair of opposite edges invariant. (*Invariant *means staying in the same place but possibly rotated in that place.)*