Here is chart with the definitions we have so far.

	commutative	noncommutative
no conditions	$2\mathbb{Z}$	
rings with unity	$\mathbb{Z}_{nm}, n, m \geq 2$	$M_n(\mathbb{R}), n \geq 2$
1 and no zero divisors	domains : $\mathbb{Z}, \mathbb{Z}[i]$	
1 and all nonzero are units	Fields : $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$	skew fields : $\mathbb{H}$

Definition 19.5. *The ring of* **quaternions** $\mathbb{H}$ *is defined to be* $\mathbb{R}^4$ *with basis* $1, i, j, k$ *and multiplication defined by:*

$$\begin{array}{ll} i^2 = j^2 = k^2 = -1 & 1i = i1 = i, \text{etc. (1 is unity)} \\ ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{array}$$

Then extend linearly. For example:

$$(a + bi)(c + dj) = ac + bci + adj + bdk$$

Theorem 19.6. $\mathbb{H}$ *is a* **skew-field** *which means a noncommutative ring with unity in which every nonzero element is a unit.*

Proof. The main point is that nonzero elements have inverses in the set:

$$(a + bi + cj + dk)^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a - bi - cj - dk)$$

□