36. Central extensions

The automorphisms θ_g are a complication that I want to ignore in this last lecture. So, I will consider the case where θ_g is always the identity mapping. This gives what is called a *central extension.*

Definition 36.1. *We say that G is a central extension of A by H if A is a subgroup of the center of G ($A \le Z(G) \Rightarrow A \trianglelefteq G$) and $G/A \cong H$.*

Example 36.2. *$SL(n, F)$ is a central extension of $Z_0 = Z \cap SL(n, F)$ by $PSL(n, F)$.*

Problem: Suppose that a semi-direct product

$$G = K \rtimes_\theta Q$$

is a central extension of K by Q. Then show that $\theta_x(k) = k$ for all $x \in Q$. Conclude that G is the direct product of the two groups:

$$G = K \times Q$$

Problem: Show that is G is a central extension of A by H then A is an abelian group.

The big question is: *Can we construct all possible central extensions of A by H up to isomorphism?*

The answer involves *transversals*, *factor sets* and *cohomology.*

36.1. transverals.

Definition 36.3. *Suppose that G is an extension of A by H. Then a* **transversal** *of H in G is defined to be subset T of G with the property that every coset of A in G contains exactly one element of T.*

Example 36.4. *Let $G = \mathbb{Z}$, $A = 3\mathbb{Z}$. A transversal of $H = \mathbb{Z}_3$ is given by $T = \{0, 5, 31\} \subset \mathbb{Z}$. Note that T is not a subgroup of $\mathbb{Z}$.*

Lemma 36.5. *The isomorphism $G/A \cong H$ induces a bijection $T \to H$.*

Proof. This is obvious. The elements of G/A are cosets of A. Each coset gA contains exactly one element $t \in T$. So, $gA = tA$. This gives a bijection $T \cong G/A$. We are given a bijection $G/A \cong H$. So, we get a bijection $T \cong H$. □

Theorem 36.6. *Let $s : H \to T$ be the bijection which sends h to the element $t \in T$ so that h corresponds to the coset tA under the isomorphism $G/A \cong H$. Then we have a bijection*

$$\phi : A \times H \to G$$

given by $\phi(a, h) = as(h)$

Proof. In lieu of a proof I will just do the example.

$$s : \mathbb{Z}_3 \to T = \{0, 5, 31\}$$

is given by $s(0) = 0, s(1) = 31, s(2) = 5$. The mapping

$$\phi : 3\mathbb{Z} \times \mathbb{Z}_3 \to \mathbb{Z}$$

is then given by

$$\phi(3n, k) = \begin{cases} 2n & \text{if } k = 0 \\ 2n + 31 & \text{if } k = 1 \\ 2n + 5 & \text{if } k = 2 \end{cases}$$

To see that this is a bijection, start with any element $m \in \mathbb{Z}$. Then k is the remainder of m after dividing by 3 and n is the quotient minus $0, 10, 1$ if $k = 0, 1, 2$ respectively. □

36.2. **factor sets.** Since we have this bijection, every element $g \in G$ is represented uniquely using two coordinates:

$$g = \phi(a, h)$$

where $a \in A, h \in H$. Now we want a formula for the coordinates of a product.

Theorem 36.7.

$$\phi(a, x)\phi(b, y) = \phi(abf(x, y), xy)$$

where f is a function

$$f : H \times H \to A$$

called a **factor set**.

I will go over the general formula for the factor set, the definition of a factor set and some examples. (Actually, I don't have time to do all this in one hour. But I can still write it.)

Proof. This is true if f is the function given by:

$$\boxed{f(x, y) = s(x)s(y)s(xy)^{-1}}$$

That follow from the formula:

$$\phi(a, x)\phi(b, y) := as(x)bs(y) = abs(x)s(y) \quad \text{since } b \text{ is central}$$

$$\phi(abf(x, y), xy) := abf(x, y)s(xy) = ab[s(x)s(y)s(xy)^{-1}]s(xy)$$

□

Theorem 36.8. *Given a group H and an abelian group A, a function $f : H \times H \to A$ is a* **factor set** *for some central extension of A by H if and only if it satisfies the following condition for all $x, y, z \in H$:*

$$f(y,z)f(xy,z)^{-1}f(x,yz)f(x,y)^{-1} = e \in A$$

This formula is called the **cocycle condition**.

Proof. First, we will show that a factor set satisfies the cocycle condition. Since $f(y,z) \in A \leq Z(G)$ we have

$$f(y,z) = s(x)f(x,y)s(x)^{-1} = s(x)s(y)s(z)s(yz)^{-1}s(x)^{-1}$$

$$f(x,yz) = s(x)s(yz)s(xyz)^{-1}$$

So,

$$f(y,z)f(x,yz) = s(x)s(y)s(z)s(xyz)^{-1}$$

Also

$$f(x,y)f(xy,z) = s(x)s(y)s(xy)^{-1}s(xy)s(z)s(xyz)^{-1} = s(x)s(y)s(z)s(xyz)^{-1}$$

So

$$f(y,z)f(x,yz) = f(x,y)f(xy,z) \tag{36.1}$$

Putting everything on one side of this equation and using the fact that A is commutative, we get the cocycle condition.

To show the converse, we need to start with a function f satisfying the cocycle condition and construct a central extension for which f is the factor set. The construction is as follows.

Let G be the set $A \times H$ with multiplication defined by

$$(a,x)(b,y) = (abf(x,y), xy)$$

This operation is associative because:

$$[(a,x)(b,y)](c,z) = (abf(x,y), xy)(c,z) = (abcf(x,y)f(xy,z), xyz)$$

$$(a,x)[(b,y)(c,z)] = (a,x)(bcf(y,z), yz) = (abcf(x,yz)f(x,yz), xyz)$$

These are equal by (36.1) which is equivalent to the cocycle condition. The operation has an identity:

$$(f(e,e)^{-1}, e)$$

This is the identity since $f(e,z) = f(e,e)$ for all $z \in H$ (just take $x = y = e$ in the cocycle condition).

Inverses are given by

$$(a,x)^{-1} = (a^{-1}f(e,e)^{-1}f(x,x^{-1})^{-1}, x^{-1})$$

If we take the transversal given by $s(x) = (e,x)$ then f will be the corresponding factor set. □

Example 36.9. *Here is an easy example of a factor set with* $H = \mathbb{Z}_n$ *and* $A = \mathbb{Z}$

$$f(x,y) = \begin{cases} 1 & \text{if } x+y \geq n \\ 0 & \text{if } x+y < n \end{cases}$$

This is the factor set for the central extension

$$\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n$$

(The notation is $K \to G \to G/K \cong Q$*.) The transversal is* $T = \{0, 1, \cdots, n-1\}$ *with* $s : \mathbb{Z}_n \to \mathbb{Z}$ *given by* $s(x) = x$*. Then*

$$f(x,y) = s(x)s(y)s(x +_n y) = x + y - (x +_n y)$$

This is either 0 *or* n *in* $n\mathbb{Z}$ *which corresponds to* 0 *or* 1 *in* $A = \mathbb{Z}$*.*

Problem: Show that if $f(x,y) = e$ then G is isomorphic to the product group $A \times H$.

36.3. group cohomology.

Definition 36.10. *An* n-**cocycle** *on a group* G *with coefficients in an additive group* A *is a function*

$$f : \underbrace{G \times G \times \cdots \times G}_{n} \to A$$

with the property that

$$f(g_1, \cdots, g_n) - f(g_0g_1, g_2, \cdots, g_n) + f(g_0, g_1g_2, g_3, \cdots, g_n) - \cdots$$
$$\cdots - (-1)^n f(g_0, g_1, \cdots, g_{n-1}g_n) + (-1)^n f(g_0, g_1, \cdots, g_{n-1}) = 0$$

for all $g_0, g_1, \cdots, g_n \in G$*.*

For example, a 2-cocycle is the same as a factor set. A 1-cocycle is a function $h : G \to A$ so that

$$h(g_1) - h(g_0g_1) + h(g_0) = 0$$

In other words, h is a homomorphism.

If h is not a homomorphism then

$$f(x,y) := h(y) - h(xy) + h(x)$$

is a 2-cocycle (factor set).

Definition 36.11. *If* $f : G^n \to A$ *is any function then the function* $\delta f : G^{n+1} \to A$ *given by the equation in the definition of an* n*-cocycle:*

$$\delta f(g_0, \cdots, g_n) = f(g_1, \cdots, g_n) - \cdots + (-1)^n f(g_0, g_1, \cdots, g_{n-1})$$

is called the **coboundary** *of* f*.*

The set of n-cycles forms an additive group under pointwise addition and the boundaries of mappings $G^{n-1} \to A$ form a subgroup. The factor group is called the nth **cohomology** of G with coefficients in A. It is denoted $H^n(G, A)$.

The main theorem about extensions is that the second cohomology group of G with coefficients in an additive group A "classifies" all central group extensions of A by G in the sense that there is a 1-1 correspondence between isomorphism classes of extensions and elements of the 2nd cohomology group $H^2(G, A)$.

If you want to learn more about this you should pick up a book on the subject of *homological algebra*. Go to
http://people.brandeis.edu/∼igusa/Math101bS07/Math101b_AwithTable.pdf
for my lecture notes on this topic (from Math 101b).

If you want to learn more about group theory, I suggest Rotman's "The Theory of Groups: an introduction." This is the 1st edition of his book which is now in the 4th edition. I recommend the 1st edition. It is the best book on group theory ever written and it sells for $12 used.

Go to
http://people.brandeis.edu/∼igusa/Math101b/Math101b.htm
for my lecture notes on a course based on this book.

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