Rules: All quizzes and practice quizzes are open book and notes. If you can't answer the question you can give partial answers or change it to get some credit. Yes, you can do that on the quiz. The closer your question is to the original question, the more credit you get.You can guess at the answer to the first part of a question to get credit for the second part as indicated. Also: Bring a calculator and/or laptop to the quiz so you can answer questions like number 3.

Write answers on a separate piece of paper or on the back of this page.

Practice Quiz I Answers

1. Suppose that X is a random variable with pdf $f(x) = 2x/\theta^2$ if $0 \le x \le \theta$ and $f(x) = 0$ for other values of x. You take a sample of size 3. Suppose you get $(X_1 = 3, X_2 = 1, X_3 = 8)$ (Always assume i.i.d.)

(1) Find the MOM estimator for θ.

You need to find $\mathbb{E}(X)$ and set it equal to $\overline{X}$ which is $12/3 = 4$ in this case.

$$\mathbb{E}(X) = \int_0^\theta x f(x)\,dx = \int_0^\theta \frac{2x^2}{\theta^2}\,dx = 2\theta/3$$

Set this equal to $\overline{X}$ and you get:

$$\hat{\theta} = \frac{3\overline{X}}{2}$$

This is the *estimator*. The data gives the *estimate* $\hat{\theta} = 6$.

(2) Is this (your answer to (1)) an unbiased estimator for θ? If not, find an unbiased estimator.

To figure out whether your estimator is unbiased, you need to take the expected value of the estimator:

$$\mathbb{E}(\hat{\theta}) = \frac{3}{2}\mathbb{E}(\overline{X}) = \frac{3}{2}\mathbb{E}(X) = \frac{3}{2}\left(\frac{2}{3}\theta\right) = \theta$$

It is always true that $\mathbb{E}(\overline{X}) = \mathbb{E}(X) = \mu$ since

$$\mathbb{E}(\overline{X}) = \mathbb{E}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\sum \mathbb{E}(X_i) = \frac{1}{n}\sum \mu = \mu$$

(3) What is the efficiency of this estimator (your answer to (1))? (Use $\hat{\theta} = \bar{X}$ if you didn't get an answer to (1).)

The efficiency is the inverse of the variance: $\text{Eff}(\hat{\theta}) = 1/\text{Var}(\hat{\theta})$. First you need the expected value of X^2:

$$\mathbb{E}(X^2) = \int_0^\theta x^2 f(x)\,dx = \int_0^\theta \frac{2x^3}{\theta^2}\,dx = \frac{\theta^2}{2}$$

So,

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{\theta^2}{2} - \frac{4\theta^2}{9} = \frac{\theta^2}{18}$$

This makes

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}\left(\frac{3}{2n}\sum X_i\right) = \frac{9}{4n^2}\sum \mathrm{Var}\, X_i = \frac{9n\theta^2}{4n^2(18)} = \frac{\theta^2}{8n}$$

So, the efficiency is

$$\mathrm{Eff}(\hat{\theta}) = \frac{1}{\mathrm{Var}(\hat{\theta})} = \frac{8n}{\theta^2}$$

Here is a new question: How does this compare with the Cramer-Rao bound?

2. Given that the mean is zero ($\mu = 0$) Show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2$$

is an unbiased estimator for σ^2.

$\mathbb{E}(X^2) = \sigma^2 + \mu^2 = \sigma^2$ since $\mu = 0$. This means that

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2) = \frac{n}{n}\sigma^2 = \sigma^2$$

So, this is an unbiased estimator of σ^2.

3. (extra credit) From a sample of size 240 from a uniform distribution you get $Y_{max} = 4000$. Find a 95% confidence interval for θ in the form

$$4000 < \theta < \hat{\theta}$$

To get a confidence interval for θ you need to write down the 95% probability range for the data Y_{max} and then solve it for θ in terms of Y_{max}. If the same size is 1 then $Y = Y_1 = Y_{max}$. The probability statement is:

$$\mathbb{P}(.05\theta < Y < \theta) = .95$$

The two inequalities, when solved for θ are:

$$.05\theta < Y \Rightarrow \theta < 20Y$$

$$Y < \theta$$

Put these together and you get the probability statement:

$$\mathbb{P}(Y < \theta < 20Y) = .95$$

When you put in the data $Y = 4000$, you get a statement about confidence instead of probability. With 95% confidence we can say that

$$4000 < \theta < 80,000$$

For the original question,

$$\mathbb{P}(p\theta < Y_{max} < \theta) = 1 - p^{240} = .95$$

So

$$p = .05^{1/240} = 0.987595$$

The inequality $p\theta < Y_{max}$ when solved for θ gives

$$\theta < \frac{1}{p} Y_{max} = 1.01256 Y_{max} = 4050.24$$

So, the 95% confidence interval for θ is

$$4000 < \theta < 4050.24$$