Practice Quiz I More

1. Suppose that X is a random variable with pdf $f(x) = 2x/\theta^2$ if $0 \le x \le \theta$ and $f(x) = 0$ for other values of x. You take a sample of size 3. Suppose you get $(X_1 = 3, X_2 = 1, X_3 = 8)$ (Always assume i.i.d.)

(1) Find the MOM estimator for θ.
(2) Is this (your answer to (1)) an unbiased estimator for θ? If not, find an unbiased estimator.
(3) What is the efficiency of this estimator (your answer to (1))? (Use $\hat{\theta} = \bar{X}$ if you didn't get an answer to (1).)

As some of you pointed out, the question should say "What is the efficiency of this estimator (your answer to (2))?" Since you can only talk about the efficiency of an unbiased estimator. (However, your answers to (1) and (2) should be the same.) The variance of the estimator is

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}\left(\frac{3}{2n}\sum X_i\right) = \frac{9}{4n^2}\sum \mathrm{Var}\, X_i = \frac{9n\theta^2}{4n^2(18)} = \frac{\theta^2}{8n}$$

Here is a new question: How does this compare with the Cramer-Rao bound? The (naive interpretation of) Fisher information is that it is given as follows:

$$\ln L(\theta) = \ln(2x) - 2\ln\theta$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = -\frac{2}{\theta}$$

$$I(\theta) = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\ln L(\theta)\right)^2\right] = \frac{4}{\theta^2}$$

The Cramer-Rao lower bound is then:

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{8n} \ge ? \frac{1}{nI(\theta)} = \frac{\theta^2}{4n}$$

In other words $\hat{\theta}$ is more effective than possible according to Cramer-Rao. Why is that?

Here is another question: What is the MLE for θ?

2. Given that the mean is 1 ($\mu = 1$) Show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - 1)^2$$

is an unbiased estimator for σ^2.

3. (extra credit) From a sample of size 1 from an exponential distribution you get $T = 5$. Find a 95% confidence interval for λ