Review for Quiz 2

1. Basic properties of $\mathbb{E}$ and Var.

(1) Given that X, Y are independent random variables calculate

$$\mathbb{E}((X+Y)^2)$$

in terms of $\mu_X, \mu_Y, \sigma_X, \sigma_Y$. These Greek letters are defined to be the mean $\mu_X = \mathbb{E}(X)$ and standard deviation $\sigma_X^2 = \mathrm{Var}(X)$ of the variables.

Since

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = \mu_X + \mu_Y$$

and

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \sigma_X^2 + \sigma_Y^2$$

we get

$$\mathbb{E}((X+Y)^2) = \mathrm{Var}(X+Y) + \mathbb{E}(X+Y)^2$$
$$= \sigma_X^2 + \sigma_Y^2 + \mu_X^2 + 2\mu_X\mu_Y + \mu_Y^2$$

(2) If $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators for θ_1, θ_2 then show that $2\hat{\theta}_1 + 3\hat{\theta}_2$ is an unbiased estimator for $2\theta_1 + 3\theta_2$.

Since expected value is linear,

$$\mathbb{E}(2\hat{\theta}_1 + 3\hat{\theta}_2) = 2\mathbb{E}(\hat{\theta}_1) + 3\mathbb{E}(\hat{\theta}_2) = 2\theta_1 + 3\theta_2$$

So, $2\hat{\theta}_1 + 3\hat{\theta}_2$ is an unbiased estimator for $2\theta_1 + 3\theta_2$

2. MLE and MOM

(1) Given that X has continuous distribution $f(x) = e^{\theta - x}$ for $x \geq \theta$ and $f(x) = 0$ for $x < \theta$ find the MOM estimator for θ.

The expected value is given by the integral

$$\int_\theta^\infty x e^{\theta - x} dx = -xe^{\theta - x} - e^{\theta - x}|_\theta^\infty = \theta + 1$$

Set this equal to $\overline{X}$ and solve for θ to get:

$$\hat{\theta}_{MOM} = \overline{X} - 1$$

Given a sample of two: $X_1 = 5, X_2 = 35$ find the MLE for θ.

The likelihood function if the probability density of getting the numbers 5 and 35:

$$L(\theta) = f(5)f(35) = e^{\theta - 5}e^{\theta - 35}$$

provided that $\theta \leq 5$. This is an increasing function of θ. So it is maximized when θ is as large as it can be which is $X_{\min} = 5$

$$\hat{\theta}_{MLE} = X_{\min} = 5$$

(2) Given that X is a discrete random variable taking only the three values $0, 1, 2$ with probability

$$\mathbb{P}(X = 0) = \theta, \quad \mathbb{P}(X = 1) = 2\theta, \quad \mathbb{P}(X = 2) = 1 - 3\theta$$

If you take a sample of $n = 4$ and get the numbers $0, 2, 0, 1$ find the MLE for θ.

This is a discrete version of the previous problem. The likelihood is the probability of getting the data that you got:

$$L(\theta) = \mathbb{P}(X_1 = 0, X_2 = 2, X_3 = 0, X_4 = 1) = \theta \cdot 2\theta \cdot \theta(1 - 3\theta) = 2\theta^3 - 6\theta^4$$

You do not have to take the natural log. (You take $\ln L(\theta)$ when it is convenient, which is usually.) So, take the derivative and set it equal to zero. Keep in mind that the maximum might be at the endpoints which are $\theta = 0, 1/3$.

$$L'(\theta) = 6\theta^2 - 24\theta^3 = 6\theta^2(1 - 4\theta)$$

This is equal to 0 when $\theta = 0, 1/4$ If you look at $L(\theta)$ or graph it on a graphing calculator you see that the maximum is at $\theta = 1/4$. So,

$$\hat{\theta}_{MLE} = \frac{1}{4}$$

3. Bias and efficiency I will do this tomorrow.

Are the estimators that you got in the previous problem unbiased? If so, find the efficiency.

4. You take a sample of 25 from an exponential distribution and get $\overline{T} = 2.3$. Find the 95% confidence interval for λ. [$\overline{T}$ is approximately normal by the CLT.]

5. Suppose that X has density function $f(x) = 2\lambda e^{-\lambda x^2} x$ if $x \geq 0$ and $f(x) = 0$ for $x < 0$. Then find the Cramér-Rao lower bound for the variance of any unbiased estimator for λ. Fisher information $I(\lambda)$ is the negative of the expected value of the second derivative of the log likelihood function since the support of the density is independent to λ.