# CATALAN NUMBERS AND EXCEPTIONAL SEQUENCES

KIYOSHI IGUSA

Contents



*Date*: September 19, 2008.

## 1. Woo's diagrams

This lecture is based on joint work with Ralf Schiffler. I also got a lot of help from Hugh Thomas and Gordana Todorov.

Ralf Schiffler and I were discussing a paper by Alexander Woo[Woo]. Woo had some drawings in his paper which we recognized as the "clusters" or "tilting objects" in the cluster category. The looked like this:

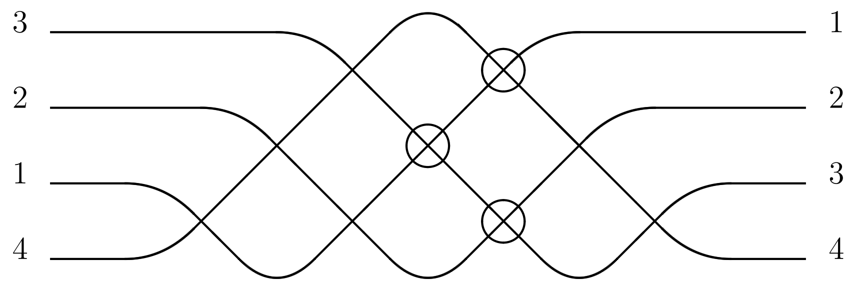



Figure 1. The ×'s are the objects of the Cluster Category. There are 9 objects here. The three circled objects form a "cluster."

Woo performed the following modification of the picture.

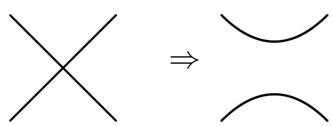



Figure 2. Do this replacement at each circled location.

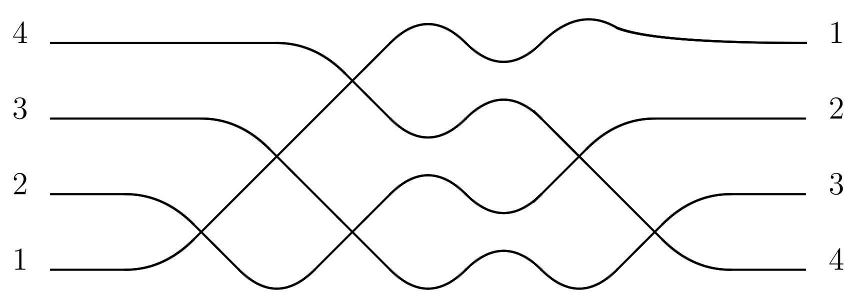



Figure 3. The result is the "longest word" $w_0 = (14)(23)$

## 2. Catalan numbers

The Catalan numbers are given by:

$$C(n) := \frac{1}{n}\binom{2n}{n-1}$$

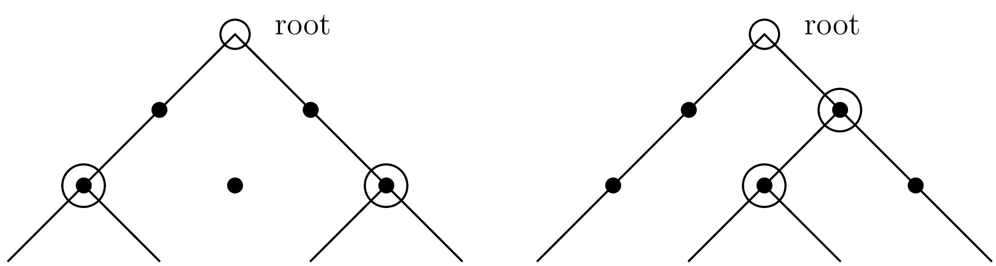
So,

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| C(n) | 1 | 2 | 5 | 14 | 42 | 132 |

$C(n)$ is the number of binary rooted trees with $n$ nodes. For the purpose of this lecture binary trees should be drawn as follows.

(1) Put the leaves in a straight line, equally spaced.
(2) Make all edges of the tree of slope 1 or -1.

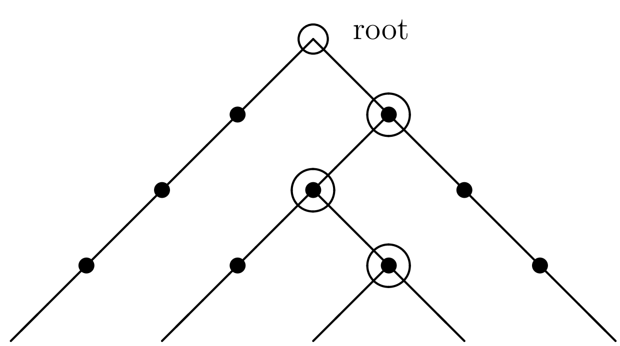
Here are two of the the $C(3) = 5$ binary trees with 4 leaves drawn according to these instructions.

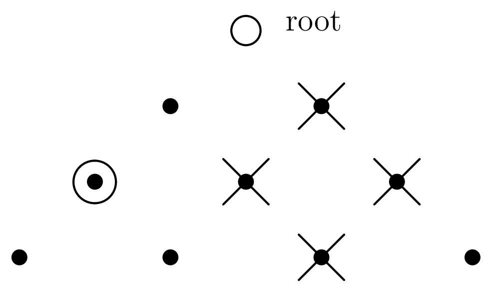



The black spots indicate the (indecomposable) objects of the "cluster category." The node at the top is called the "root." The other nodes form a "cluster" with $n-1$ objects.

**Theorem 2.1.** *Each binary tree is determined by its set of nodes.*

For example, if we take the cluster in the Woo diagram we get:

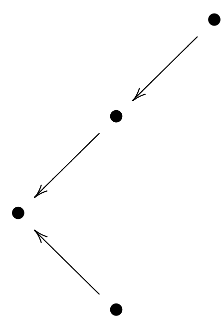





**Exercise 2.2.** : Show that, if there is a node at the circled position, there cannot be a node at any of the positions marked "×"

## 3. Representations of quivers

A quiver is a directed graph $Q$. We assume that the quiver has no oriented cycles. Let's take the following really simple example called $A_4$.



The arrows should all point left.

**Definition 3.1.** A *representation* of the quiver $Q$ is given by putting a vector space $V_i$ at each vertex and a linear map $V_a : V_i \to V_j$ on each arrow $a : i \to j$ in the quiver.

**Theorem 3.2.** *Representations are the same as modules over a ring $KQ$. If $Q$ is a Dynkin diagram, the isomorphism classes of indecomposable representations are in 1-1 correspondence with the positive roots of the root system of $Q$.*

This means that the indecomposables are

$$M(\alpha)\ , M(\beta)\ , M(\gamma)\ , \cdots \quad \alpha, \beta, \gamma, \cdots \in \Phi_+$$

and $\dim M_i(\beta) = \beta_i$ (the coordinates of $\beta$ are the dimensions of the vectors spaces $M_i(\beta)$).

**Definition 3.3.** A (complete) exceptional sequence is a sequence of representations

$$M(\beta_1), M(\beta_2), \cdots, M(\beta_n)$$

(where $n$ is the number of vertices of $Q$) so that

$$\mathrm{Hom}(M(\beta_j), M(\beta_i)) = 0 = \mathrm{Ext}(M(\beta_j), M(\beta_i))$$

for all $i < j$.

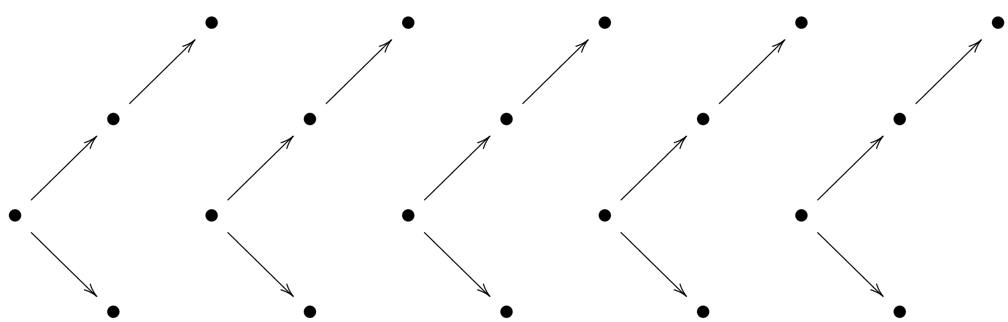
How do I explain this to my students in Math 47a?

Simple. We convert these into number which can be calculated by a very simple formula using the Auslander-Reiten quiver which is also very simple. The numbers are:

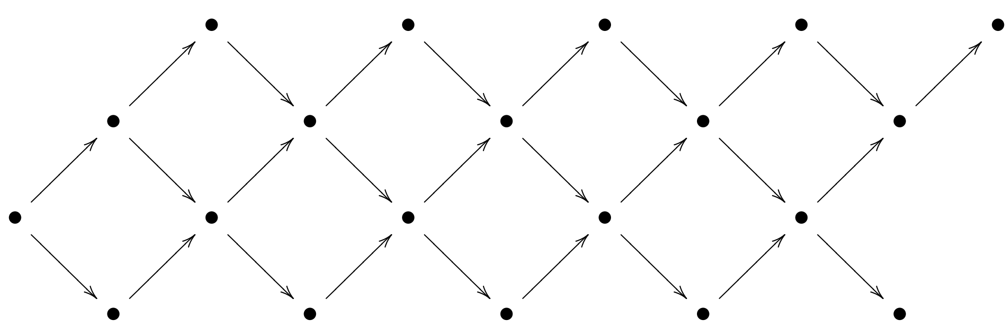
$$hom(\alpha, \beta) = \dim \mathrm{Hom}(M(\alpha), M(\beta))$$

$$ext(\alpha, \beta) = \dim \mathrm{Ext}(M(\alpha), M(\beta))$$

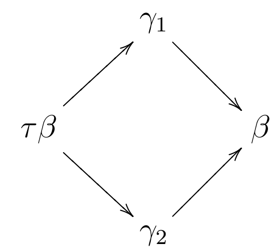
The *Auslander-Reiten quiver* is given taking an infinite number of copies of $Q$ with arrows reversed (going to the right)



plus an arrow from $i \to j$ on the next copy of $j$ for each arrow in $Q$:



This big graph is called the Auslander-Reiten quiver of the "derived category" It has positive and negative roots as vertices. The shifting operation operation to the left is called Auslander-Reiten translation and written $\tau$:




Here is the formula for $hom(\alpha, \beta)$ for a fixed $\alpha$ and variable $\beta$:

$$hom(\alpha, \alpha) = 1$$

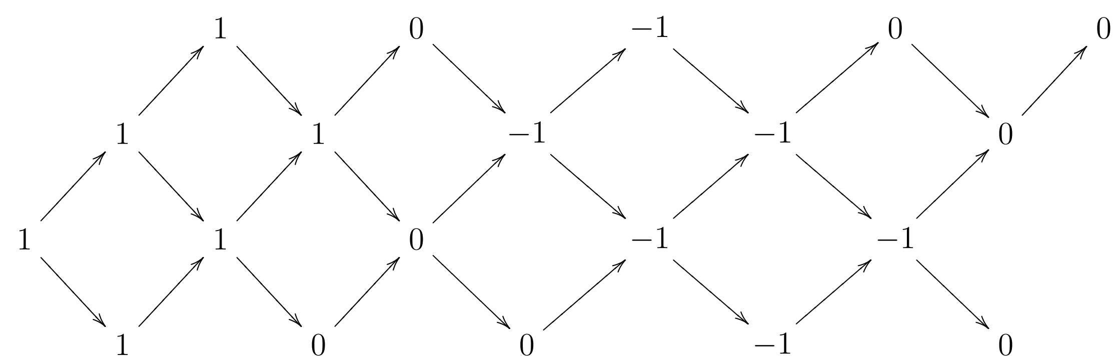
One important point is that there are no homomorphism going to the left. Thus:

$$hom(\alpha, x) = 0 \quad \text{if } x \text{ is to the left of } \alpha.$$

For $\beta$ on the right there is a recursive formula:

$$hom(\alpha, \beta) = hom(\alpha, \gamma_1) + hom(\alpha, \gamma_2) - hom(\alpha, \tau\beta)$$

for $\alpha \neq \beta$. For type $A_n$ this gives regular patterns of 1's and $-1$'s:




Of course the $-1$'s don't make much sense. These locations are negative roots which represent *virtual or shifted representations* "$M(\beta)[1]$."

To calculate $ext(\alpha, \beta)$ we use Serre duality:

$$ext(\beta, \alpha) = hom(\tau^{-1}\alpha, \beta) = hom(\alpha, \tau\beta)$$

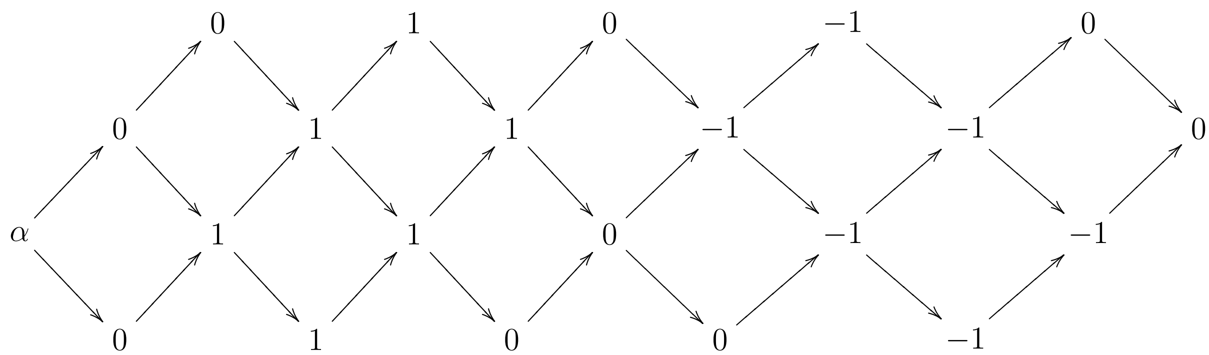
which is the numerical version of

**Theorem 3.4.** *There is a natural isomorphism:*

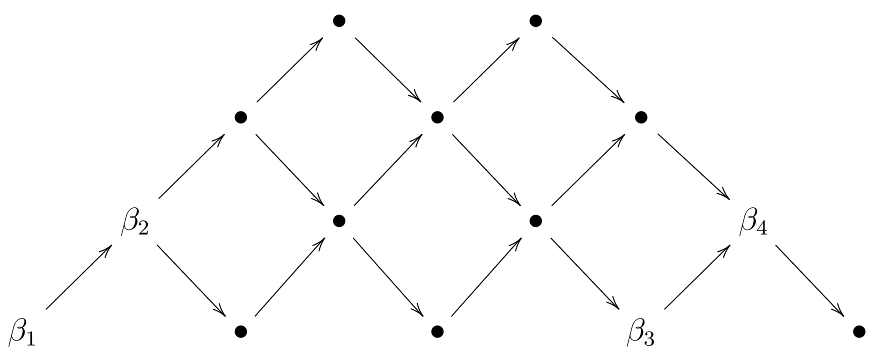
$$\mathrm{Ext}(M(\beta), M(\alpha)) \cong D\,\mathrm{Hom}(M(\tau^{-1}\alpha), M(\beta))$$

*where* $D = \mathrm{Hom}(-, K)$ *is vector space dual.*

For a fixed $\alpha$, the numbers $ext(\beta, \alpha)$ are therefore given by taking the *hom* numbers and shifting them one space to the right:




**Exercise 3.5.** Show that the sequence $\beta_1, \beta_2, \beta_3, \beta_4$ in the diagram below is an exceptional sequence.



## 4. Statement of theorem

We only discussed type $A_n$. The theorem here is the following.

Suppose that $\beta_1, \beta_2, \cdots, \beta_n$ are positive roots of the root system $A_n$ in canonical order (i.e., from left to right in the picture). If you did the exercises it should be clear to you that the following are equivalent.

(1) $M(\beta_1), M(\beta_2), \cdots, M(\beta_n)$ is an exceptional sequence.
(2) The positions of $\beta_1, \cdots, \beta_n$ are the locations of the non-root nodes of a binary tree with $n+2$ leaves.

Woo's theorem says that these are equivalent to the following condition:

**Theorem 4.1.** *$\beta_1, \beta_2, \cdots, \beta_n$ is an exceptional sequence if and only if the product of the corresponding transpositions is equal to the "Coxeter" permutation $(n+1, 1, 2, \cdots, n) \in S_{n+1} = W(A_n)$*

What Woo actually did is to *count* the number of elements in the second set. He showed it is the Catalan number. But everybody knows that the first set has a Catalan number of elements and the first set is a subset of the second set by the following theorem of Crawley-Boevey which works for arbitrary quivers.

This can be generalized to arbitrary reflection groups by the following two theorems.

**Theorem 4.2** (Crawley-Boevey[CB93])**.** *If $M(\beta_1), M(\beta_2), \cdots, M(\beta_n)$ is an exceptional sequence then the product of the corresponding reflections is the Coxeter element for an arbitrary quiver (i.e., for any simply-laced crystallographic reflection group).*

Ringel extended this to the nonsimply-laced case.

Ralf and I proved the converse:

**Theorem 4.3.** *If $M(\beta_1), M(\beta_2), \cdots, M(\beta_n)$ is an exceptional sequence if and only if the product of the corresponding reflections is the Coxeter element of $W$ in any simply-laced crystallographic reflection group.*

This was known in each finite case by various people and was extended to the affine case by Ingalls and Thomas[IT].

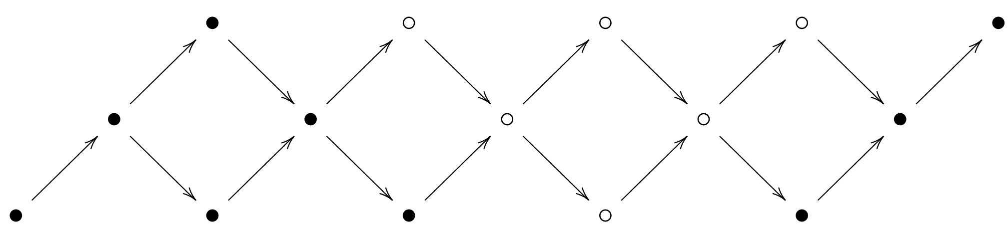
## 5. Extension to Artin groups

The question or conjecture that I stated, in the form of a drawing, on the first day of class is the following.

**Question 5.1.** *Does the $m$-ified version of this theorem hold for the braid group?*

How did I explain or try to explain this question to my undergraduate class on the first day of class?

The cluster category is given by taking a triangle and adding a line of points on the side to form a bigger triangle with the top point missing. The $m$-cluster category is given by taking $m$ copies of the triangle with every second copy turned upside-down so that they fit together and then adding a line of points. So, for $m = 2$ we get:



Here the black dots are positive roots and the white dots are negative roots.

A cluster in the $m$-cluster category is a set of points $\beta_1, \beta_2, \cdots, \beta_n$ (numbered from left to right in the picture) which satisfies either of the following equivalent conditions.

(1) $\beta_1, \beta_2, \cdots, \beta_n$ is an exceptional sequence.
(2) When the points are deleted we get the longest word $w_0$ for $m$ odd and the identity for $m$ even.

Here is an example.

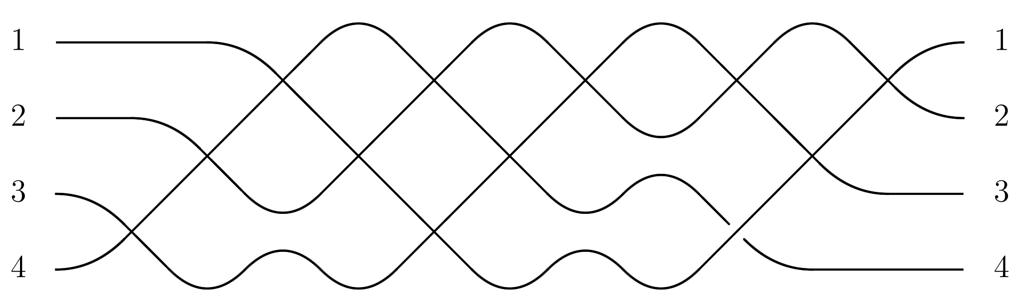



Figure 4. The three missing points form an exceptional sequence ⇔ the permutation is the identity.

Here is the three dimensional version of the same picture:

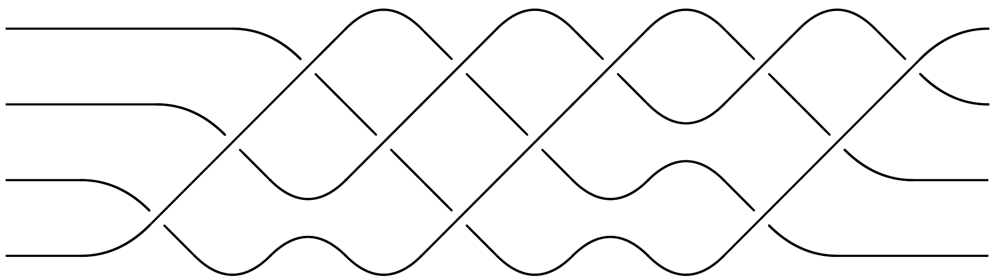


FIGURE 5. This is the braid corresponding to the exceptional sequence (32)(41)(42). This braid is equal to $\Delta^2$ which is the 360° lefthand twist. (Since the braid is sideways, what looks like a right-handed twist is actually left-handed. So, these picture are vertical mirror images of the pictures in [BKL98].)

We know that, if these strings are allowed to cross, they can be straightened out. This is the gist of Crawley-Boevey's theorem and its converse as applied to the "wire diagrams" But since all of the crossing are "positive" this braid is obviously twisted. The question was whether the "mutations" of the picture were braid moves. The *mutations* of the configuration are given by moving two strands. If strands $i, j$ cross at a point $\beta^*$ and don't cross at the point $\beta$ (part of the $m$-cluster) then we can redraw the picture so that they cross at $\beta$ but not at $\beta^*$. This removes $\beta$ from the cluster and replaces it with $\beta^*$.

We know that mutation gives the same permutation. In fact, that is obvious. But do they give the same braid? The answer is "yes" and the proof is given by the following picture and the fact that mutation is transitive on the set of all clusters.

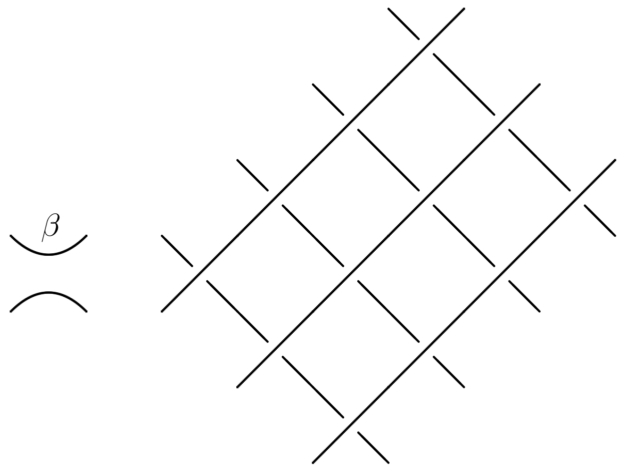



FIGURE 6. To the right of the object $\beta$ there is a rectangular region of incompatible modules. So there are crossings at these positions for any $m$-cluster.

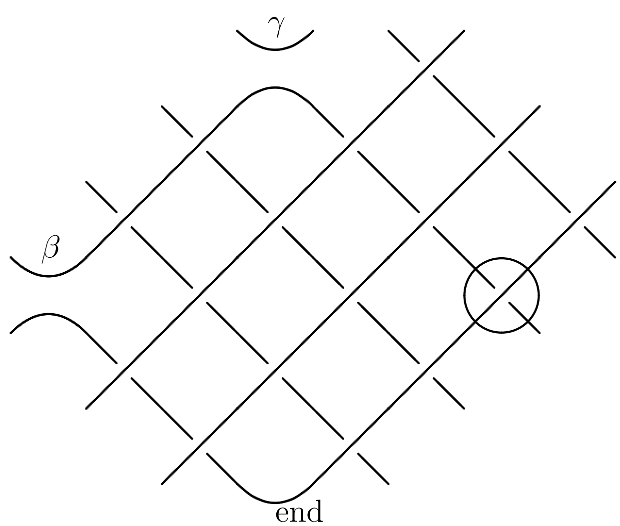


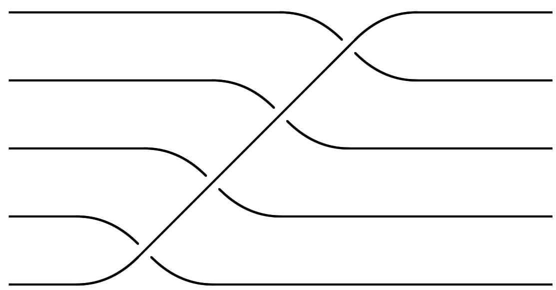

FIGURE 7. Somewhere along the top left and the bottom left there are either gaps or you come to the end of the graph. By looking at the picture you see that the gap at $\beta$ can be exchanged for the crossing (circled) by an isotopy of the graph.

This proves or rather reproves the following theorem.

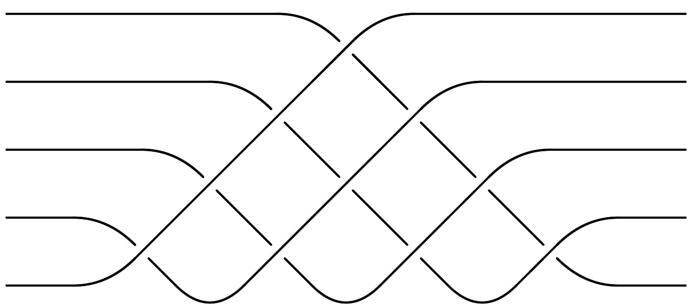
**Theorem 5.2.** *A product of a sequence of $n$ generators of the braid group (as given by Birman, Ko and Lee*[BKL98] *is equal to* $\delta' = \sigma_1\sigma_2\cdots\sigma_n$ *if and only if it maps to the Coxeter element of the symmetric group.*

In fact, the analogous statement is know to be true for all finite cases[Dig]. (If my understanding of French serves me.)

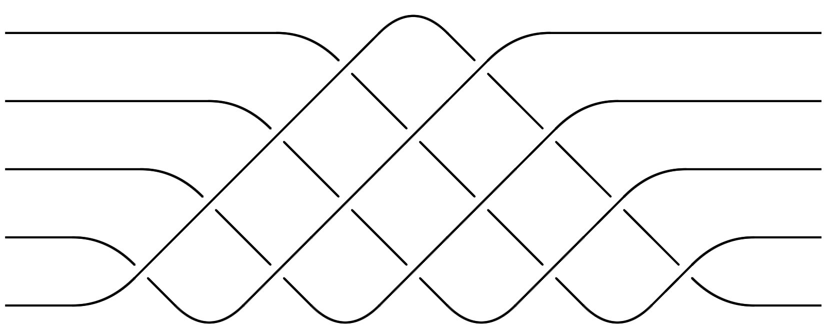
$\delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$



$\Delta =$



$\delta\Delta = \Delta\delta'$



## 6. Comments

Birman, Ko and Lee[BKL98] showed that braids can be put into canonical form using $\delta$ and "canonical factors". This work had a big impact. For me it was the word "Catalan" They showed that there are a Catalan number of canonical factors. They listed the 14 canonical factors for $n = 3$ and you can see these are exactly the reverse exceptional sequences.

Drew Armstrong wrote a PhD thesis[Arm] about the combinatorics of $m$-clusters and related topics. Armstrong's thesis is filled with detailed historical comments some of which I read to the class.

## References

[Arm] Drew Armstong, *Generalized noncrossing partitions and combinatorics of coxeter groups*, arXiv:math/0611106, to appear in Memoires of AMS.

[BKL98] Joan Birman, Ki Hyoung Ko, and Sang Jin Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. **139** (1998), no. 2, 322–353.

[CB93] William Crawley-Boevey, *Exceptional sequences of representations of quivers [ MR1206935 (94c:16017)]*, Representations of algebras (Ottawa, ON, 1992), CMS Conf. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1993, pp. 117–124.

[Dig] François Digne, *Présentations duales des groupes de tresses de type affine* $\tilde{a}$, arXiv:math/0402116.

[IT] Colin Ingalls and Hugh Thomas, *Noncrossing partitions and representations of quivers*, arXiv:math/0612219.

[Woo] Alexander Woo, *Catalan numbers and schubert polynomials for* $w = 1(n+1)...2$, arXiv:math/0407160.