1. Problem Set 1

The goal is to try all the problems and do half of them.

1.1. Suppose that σ is a permutation on n with k cycles. Then show that the composition $(12)\sigma$ has $k, k-1$ or $k+1$ cycles. Give an example of each.

1.2. Suppose that a, b are permutations and x, y, z are also permutations (or: take a, b, x, y, z to be elements of any group G). Suppose that

$$xab = ayb = abz.$$

(1) Show that x, y, z are conjugate to each other.
(2) When are x, y, z all different? Give an example when they are different and when two of them are the same.

1.3. The permutation (13) can be written as a product of simple transposition $(i, i+1)$ in two different ways. Draw a picture and find these two expressions. Can you prove that there are no other expressions for (13) with the same length?

1.4. Same question for (14).

1.5. Suppose that α, β are two vectors in $\mathbb{R}^3$ and θ is the angle between them. Let r_α, r_β be the reflections along α, β respectively.

(1) Describe the compositions $r_\alpha \circ r_\beta$ and $r_\beta \circ r_\alpha$ geometrically. (They are rotations. What is the axis of rotation and what angle do they rotate?) Draw a picture.
(2) Conclude that, for a root system in $\mathbb{R}^2$, the lengths of the roots alternate between two lengths as you go around a circle.
(3) When do r_α, r_β commute?

1.6. The *order* of a permutation σ is the smallest positive integer m so that $\sigma^m = e$. For example, the order of the element

$$\sigma = (12)(345)$$

is 6. The formula is: the order of σ is the least common multiple of the lengths of the cycles in σ. Why is this true?

1.7. Find an interpretation for the areas under a binary tree. Do all possible areas occur? Do the areas determine the tree?

(The way it is drawn here, the numbers are the areas plus $\frac{1}{2}$.)