1. Short Answers to Problem Set 1

You could write a long answer to one for midterm paper.

1.1. Suppose that σ is a permutation on n with k cycles. Then show that the composition $(12)\sigma$ has $k, k-1$ or $k+1$ cycles. Give an example of each.

There are 6 cases.

(1) Neither 1 nor 2 is in one of the cycles of σ.
(2) One of them is in a cycle of σ, the other isn't.
(3) $1, 2$ are in different cycles of σ.
(4) (12) is one of the cycles of σ.
(5) 12 are in the same cycle of σ and they are consecutive (12 or 21) and the cycle has more than two letters: $(12x..)$ or $(21x..)$
(6) 12 are in the same cycle of σ but they are not consecutive: $(1xxx2yyy)$.

Let's take the second to the last case: We take the case of $(12x...)$ the other case is the same. Thus:

$$\sigma = (12x_1 \cdots x_n)(k-1 \text{ other cycles})$$

where $k \geq 1$. This makes

$$(12)\sigma = (2x_1 \cdots x_n)(k-1 \text{ other cycles})$$

So, $(12)\sigma$ has k cycles in this case.

Here is an example of case (2): $\sigma = (234)(56)$ with $k = 2$. Then

$$(12)\sigma = (1234)(56) \quad \text{with} \quad k = 2.$$

k stays the same in this case.

1.2. Suppose that a, b are permutations and x, y, z are also permutations (or: take a, b, x, y, z to be elements of any group G). Suppose that

$$xab = ayb = abz.$$

(1) Show that x, y, z are conjugate to each other.
(2) When are x, y, z all different? Give an example when they are different and when two of them are the same.

(1) This just follows from the definition.
(2) x, y are different when a, y do not commute. y, z are different when b, y do not commute. x, z are different when x does not commute with ab. But it would be good to have an answer

which involves only one letter. So, I want to write this in terms of y:

$$x = aya^{-1} = z = b^{-1}zb \quad \Leftrightarrow \quad baya^{-1}b^{-1}$$

So, one nice answer would be: x, y, z are all different if y does not commute with a, b or with ba.

1.3. The permutation (13) can be written as a product of simple transposition $(i, i+1)$ in two different ways. Draw a picture and find these two expressions. Can you prove that there are no other expressions for (13) with the same length?

The answer is

$$(13) = (12)(23)(12) = (23)(12)(23)$$

There is not other way to write (13) as a product of 3 simple transpositions. You can prove this in three steps.

(1) The first step is to show that you need at least three simple transpositions.
(2) Show that the other letters $4, 5, 6$, etc. cannot be permuted (since you would then need to permute them back).
(3) There are only two simple transpositions which don't involve the higher numbers. They are (12) and (23). You can't use the same one twice in a row since they would cancel.

1.4. Same question for (14).

I probably meant "analogous question" This would be: How many ways can you write (14) as a product of simple transpositions. The answer is six. The procedure to prove this might be as follows.

(1) Show that you need at least 5 simple transpositions.
(2) Show that the higher letters: $5, 6, 7$, etc. cannot be moved.
(3) Show that, in the picture, lines 2 and 3 cannot cross.
(4) The lines 1 and 4 cross either above line 2, below line 3 or between these two lines.

1.5. Suppose that α, β are two vectors in $\mathbb{R}^3$ and θ is the angle between them. Let r_α, r_β be the reflections along α, β respectively.

(1) Describe the compositions $r_\alpha \circ r_\beta$ and $r_\beta \circ r_\alpha$ geometrically. (They are rotations. What is the axis of rotation and what angle do they rotate?) Draw a picture.
(2) Conclude that, for a root system in $\mathbb{R}^2$, the lengths of the roots alternate between two lengths as you go around a circle.
(3) When do r_α, r_β commute?

The vectors α, β span a plane. The axis of rotation will be the line through the origin which is perpendicular to this plane. The angle of rotation is 2θ and it goes from the first vector through the second. Since the double reflection of α lands on the other side of β, the two vectors hop scotch over each other making an alternating pattern. The two reflections commute when the angle between them is ninety degrees.

1.6. The *order* of a permutation σ is the smallest positive integer m so that $\sigma^m = e$. For example, the order of the element

$$\sigma = (12)(345)$$

is 6. The formula is: the order of σ is the least common multiple of the lengths of the cycles in σ. Why is this true?

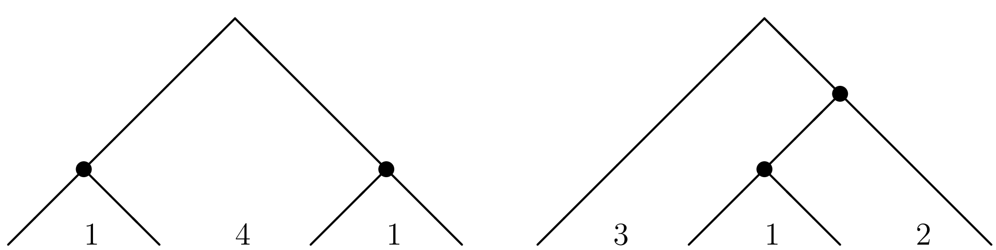
If σ is a product of cycles $\sigma = \prod C_i$ with order n_i then

$$\sigma^m = \prod C_i^m$$

which is equal to e if and only if each $C_i^m = e$ iff the order of each C_i divides m iff the least common multiple of these orders divides m.

1.7. Find an interpretation for the areas under a binary tree. Do all possible areas occur? Do the areas determine the tree?

(The way it is drawn here, the numbers are the areas plus $\frac{1}{2}$.)



The numbers that occur must be products ab of two positive integers with sum $a + b \leq n + 1$. Every number of this kind will occur in some tree with n nodes. The sum of the numbers is equal to the triangle number $n(n+1)/2$ since this is the total area under the triangle. These numbers determine the tree.

Problem: Is there an easy way to determine which sequences of allowable numbers you can get?