2. Problem Set 2

The goal is to try all the problems and do half of them.

2.1. Suppose that $\mathcal{C}$ is a category with 2 objects A, B so that there is only one morphism $A \to A$, only one morphism $A \to B$ and only one morphism $B \to B$. Then how many morphisms can there be from $B \to A$? How can you tell whether a number is possible?

2.2. Noncrossing partitions can be rotated. Rooted binary trees can also be rotated by attaching an edge at the root, deleting the edge attached to a leaf, then rotating the new 2-valent node to the top. Do these rotations correspond? (Does the bijection between rooted binary trees and noncrossing partitions take the rotated tree to the rotated partition?) First do some examples of rotations to see if this seems to be true. (Or if you know why it is true or not true, do some examples to illustrate this.)

2.3. In 1791 Fuss showed that the number of ways to cut up a convex polygon P with $n + 2$ sides into triangles with corners at the corners of P is given by the Catalan number $C(n)$. Show that this is true by finding a bijection from the set of triangulations of P to the set of rooted binary trees. Hint: Put one side of P at the top. Put one leaf at each of the other sides. Place a node in the center of each triangle. Connect nodes in adjacent triangles. Connect each leaf to the node in the center of the triangle adjacent to that side. Do an example, draw a picture. What is the inverse process?

2.4. If you have a binary rooted tree, you can take its mirror image. (Hold the paper up to a mirror.) The corresponding Dyck path also changes in some mysterious way: For example $LLRRLR$ becomes $LLRLRR$. Do more example to figure out the operation on the Dyck path given by mirror image of the binary tree.

2.5. Suppose that a Dyck path is reversed, i.e., write it backwards and switch L with R: This would make $LLRRLR$ into $LRLLRR$.

(1) Show that the result is still a Dyck path.
(2) Describe what happens to the binary tree.

As always, you should do example to see if you can spot the pattern.

2.6. A *mutation* of a binary tree is given by changing the position of one node.

(1) What does a mutation do to the Dyck path?
(2) How many mutations will take $LLLLLRRRRR$ to $LRLRLRLRLR$?