2. Problem Set 2: hints

The goal is to try all the problems and do half of them.

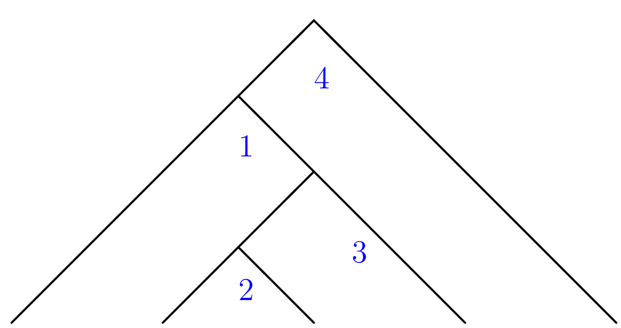
2.1. Suppose that $\mathcal{C}$ is a category with 2 objects A, B so that there is only one morphism $A \to A$, only one morphism $A \to B$ and only one morphism $B \to B$. Then how many morphisms can there be from $B \to A$? How can you tell whether a number is possible?

Something very strange happens if there are two arrows

$$g, h : B \to A$$

2.2. Noncrossing partitions can be rotated. Rooted binary trees can also be rotated by attaching an edge at the root, deleting the edge attached to a leaf, then rotating the new 2-valent node to the top. Do these rotations correspond? (Does the bijection between rooted binary trees and noncrossing partitions take the rotated tree to the rotated partition?) First do some examples of rotations to see if this seems to be true. (Or if you know why it is true or not true, do some examples to illustrate this.)

We talked about the bijection from rooted binary trees to noncrossing partitions today (10/15/08). You take a walk around the tree, going counterclockwise starting at the root. You get a picture that looks like a "glove" for the tree. You number the right edges 1,2,3 when you first come to them, which will be when you go down these edges. Then the parts of the noncrossing partition consists of the numbers on each straight line of slope -1.



This gives (13)(2)(4).

2.3. In 1791 Fuss showed that the number of ways to cut up a convex polygon P with $n+2$ sides into triangles with corners at the corners of P is given by the Catalan number $C(n)$. Show that this is true by finding a bijection from the set of triangulations of P to the set of rooted binary trees. Hint: Put one side of P at the top. Put one leaf at each of the other sides. Place a node in the center of each triangle. Connect nodes in adjacent triangles. Connect each leaf to the node in the center of the triangle adjacent to that side. Do an example, draw a picture. What is the inverse process?

Draw the binary tree with the leaves in a circle.

2.4. If you have a binary rooted tree, you can take its mirror image. (Hold the paper up to a mirror.) The corresponding Dyck path also changes in some mysterious way: For example $LLRRLR$ becomes $LLRLRR$. Do more example to figure out the operation on the Dyck path given by mirror image of the binary tree.

This is a challenge (1).

2.5. Suppose that a Dyck path is reversed, i.e., write it backwards and switch L with R: This would make $LLRRLR$ into $LRLLRR$.

(1) Show that the result is still a Dyck path. This is easy. Just cut it any point and count the number of L's and R's.
(2) Describe what happens to the binary tree. Another challenge (2)

As always, you should do example to see if you can spot the pattern.

2.6. A *mutation* of a binary tree is given by changing the position of one node.

(1) What does a mutation do to the Dyck path? Challenge (3)
(2) How many mutations will take $LLLLLRRRRR$ to $LRLRLRLRLR$?

Challenges (1),(2),(3) are related. The answer involves "jumping over" Dyck subpaths. For example, if we have $LLRLLLRLRRR$ then the red part is a Dyck subpath and the blue part jumps around the red part in a mutation to get $LLLLLRLRRRR$