MATH 47A FALL 2008
INTRO TO MATH RESEARCH

KIYOSHI IGUSA

Contents

Date: September 3, 2008.

1. Permutations or: group theory in 15 minutes

For those of you who already took a course in group theory, you probably learned about "abstract groups" which are sets with binary operations satisfying a list of conditions. I want to talk about permutations. Those who already know group theory can think about the question:

Why is every group isomorphic to a permutation group?

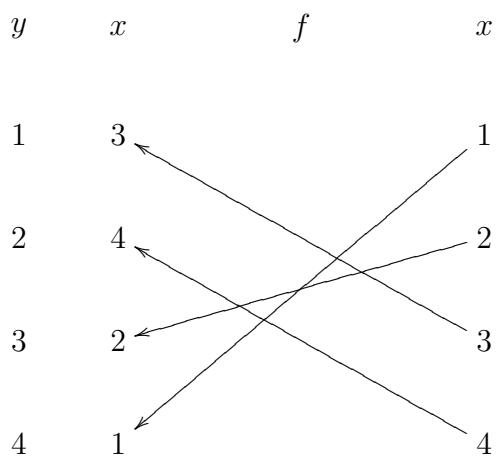
Definition 1.1. A *permutation* on a set X is a bijection $f : X \to X$. Recall that a *bijection* is a mapping which is

(1) 1-1 (injective) and
(2) onto (surjective).

If $X = \{1, 2, \cdots, n\}$, a permutation of X is called a *permutation on n letters*. The set of permutations of X will be denoted by $Perm(X)$.

I discussed three notations for permutations:

(1) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ means $\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 2, \sigma(4) = 1$ or $\sigma(x) = y$ where x is given in the first row and y is given in the second row.
(2) (cycle form) $\sigma = (1324)$. This means σ sends 1 to 3, 3 to 2, 2 to 4 and 4 to 1: $1 \to 3 \to 2 \to 4 \to 1$.
(3) (graphic notation). Here you put the numbers $x = 1, \cdots, n$ vertically on the right, put $y = 1, \cdots, n$ vertically on the left and connect each x to each $y = f(x)$. For example, if $f = (1423)$, you connect 1 on the right to $f(1) = 4$ on the left with a straight line, etc. In cycle notation, $f = (1423)$.



Definition 1.2. If $f, g \in Perm(X)$, $fg = f \circ g$ is the *composition* of f and g. This is the permutation defined by

$$fg(x) = f(g(x))$$

It means you do g first and then f.

Question: If $g = (34)$ what is gf? Write the answer in all three notations and demonstrate the composition in the notation.

(1) Stack up f, g, putting the first operation f on top and the second g underneath. Then cross out the second line:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

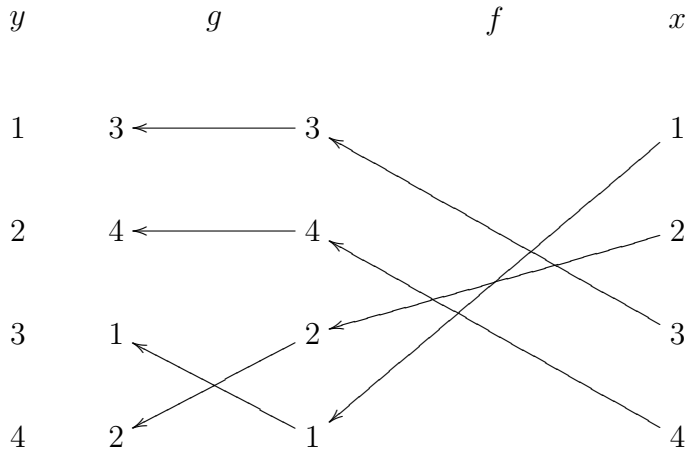
(2) Write the cycles next to each other and compute the image of each x by applying the cycles one at a time going from right to left:

$$gf = (34)(1423) = (13)(24)$$

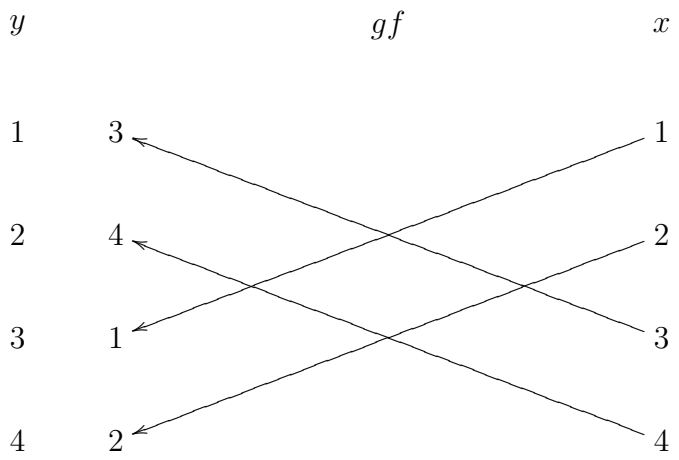
for example:

$$gf(1) =_3 (34)_4(1423)_1 = 3$$

(3) Draw the diagrams next to each other, putting the first permutation f on the right:



Note that $g = (34)$ switches the "letters" in *locations* 3 and 4. We discussed the fact that two of the crossings cancel when we redraw the picture:



Then we discussed the number of crossings.