2. Transpositions

On Day 2, I tried to formalize things we talked about at the end of Day 1, namely, in what way do the crossings in the diagram give transpositions and: What is the "longest word" ?

Definition 2.1. A *transposition* is a 2-cycle (ab). For example

$$(14) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

is a transposition. A *simple transposition* (also called a *simple reflection*) is a transposition of two consecutive letters:

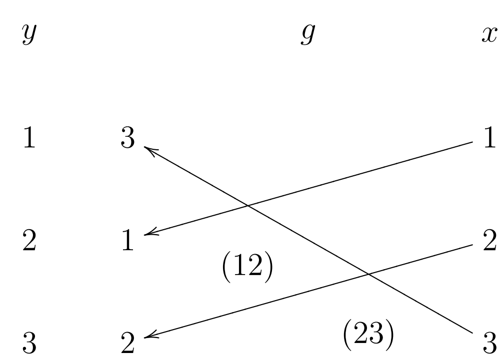
$$s_i := (i, i+1)$$

In the group of permutations of $n+1$ letters there are n simple reflections: $s_1 = (12), s_2 = (23), \cdots, s_n = (n, n+1)$.

We talked about the following vaguely phrased theorem.

Theorem 2.2. *A permutation f can be written as a product of simple reflections, one for every crossing in the diagram for f.*

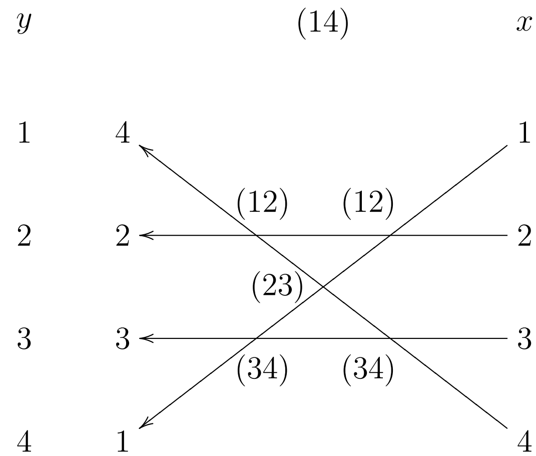
I drew a bunch of diagrams to illustrate this. The first was:



The crossing on the right is $s_2 = (23)$. The one on the left if $s_1 = (12)$. Reading these from right to left we get:

$$g = (123) = (12)(23) = s_1 s_2.$$

When two crossing lie above each other (so that you don't know which one is on the left and which is on the right) you can write them in either order. For example:



This gives:

$$(14) = (12)(34)(23)(12)(34) = s_1 s_3 s_2 s_1 s_3$$

But the crossings labeled $s_1 = (12)$ and $s_3 = (34)$ lie above each other. So we can write them in either order:

$$(14) = s_3 s_1 s_2 s_3 s_1$$

In general, the rule is

$$s_i s_j = s_j s_i \text{ if } |j - i| \geq 2$$

Definition 2.3. Suppose that f is a permutation of the letters $1, 2, \cdots, n+1$. Then the *length* $\ell(f)$ of f is defined to be the number of pairs of integers i, j so that

(1) $1 \leq i < j \leq n+1$
(2) $f(i) > f(j)$.

In other words, it is the number of pairs of numbers whose order is switched by f.

Corollary 2.4. *Any permutation f can be written as a product of $\ell(f)$ simple reflections.*

Proof. Draw the diagram of the permutation. Move the lines slightly up or down so that there are no triple crossings of lines. Lines i and j will cross if and only if the are in one order on the right ($i < j$) and in the other order on the left ($f(i) > f(j)$). Therefore, the number of crossings is equal to $\ell(f)$ as defined above. The theorem says that each crossing gives one simple reflection and that f is a product of those simple reflections. Therefore, f is a product of $\ell(f)$ reflections. □

How large can $\ell(f)$ be?

The maximum possible number occurs when every single pair of integers $i < j$ gets reversed. This is the permutation called w_0:

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n+1 \\ n+1 & n & n-1 & \cdots & 1 \end{pmatrix}$$

The length of w_0 is the number of all pairs $1 \le i < j \le n+1$ which is

$$\ell(w_0) = \binom{n+1}{2} = \frac{n(n+1)}{2} = 1 + 2 + \cdots + n$$

This is a *triangle number*. As Tri pointed out in class, w_0 can be written as a product of n s_1's, $n-1$ s_2's, etc. For example, for $n = 4$,

$$w_0 = \begin{matrix} & & & s_4 & & & \\ & & s_3 & & s_3 & & \\ & s_2 & & s_2 & & s_2 & \\ s_1 & & s_1 & & s_1 & & s_1 \end{matrix}$$

$$= s_1 s_2 s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_1$$

If we want to draw the diagram with 1 at the top and $n+1 = 5$ at the bottom, we could also write:

$$w_0 = \begin{matrix} & & & s_1 & & & \\ & & s_2 & & s_2 & & \\ & s_3 & & s_3 & & s_3 & \\ s_4 & & s_4 & & s_4 & & s_4 \end{matrix}$$

In this method the simple reflection s_i occurs i times. And this is the method I used on Day 0. There are many different ways to write w_0 as a product of $\binom{n+1}{2}$ simple reflections. We will study this in detail later.