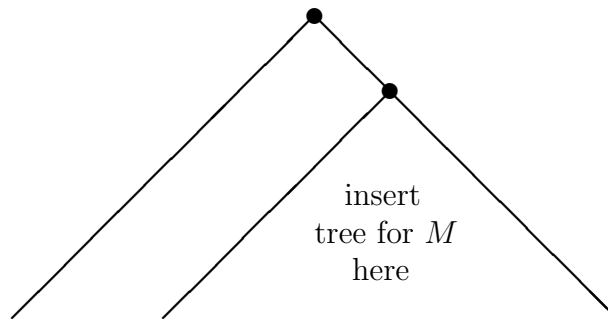


But the original Dyck path is $LABR$. So, LA is a beginning of the original Dyck path. By assumption, this word has more L 's than R 's. So, if we remove one L we have at least as many L 's as R 's. So, the first part A of the middle word has at least as many L 's as R 's. Therefore the middle word is a Dyck path. \square

In case 2 the Dyck path is LMR where M is the middle part which is a shorter Dyck path. The algorithm is: Construct the binary tree for this middle word, then put it on the right underneath the root:



5.3.3. *the can of gas.* If forgot to mention the “can of gas” analogy. In this interpretation you have a car which has no gas to start with. You have to go n miles and on the road there are n cans of gasoline. Each can has enough gas to go one mile and they are located at milestones (at 0 miles, 1 mile, 2 miles, etc.)

In order to get started you need a can of gas at the beginning. After going one mile you need to find another can of gas. In the Dyck path, the L 's are cans of gas and the R are miles. So, for example, in the Dyck path $LRLRLRR$ you get one can of gas at the beginning, then you drive one mile and find two more cans of gas. You drive one more mile and you find the fourth can of gas which is enough to get you two more miles to the finish.

If you take $LRLRLLL$ you will run out of gas after 2 miles. At any point in time you need enough gas to get to the next milestone. In other words, you need at least as many cans of gas (the L 's) as the number of miles (R 's) that you travel.

5.4. **reflection principle.** Today we used the reflection principle to prove that there number of Dyck paths is given by the Catalan number.

Theorem 5.18. *The number of Dyck paths of length $2n$ is equal to*

$$C(n) = \frac{1}{n+1} \binom{2n}{n}.$$

Proof. A Dyck path is the same as a random walk which stays on or above the y -axis and starts and ends at $y = 0$ in $2n$ steps. The total number of random walks which start and end at $y = 0$ in $2n$ steps is

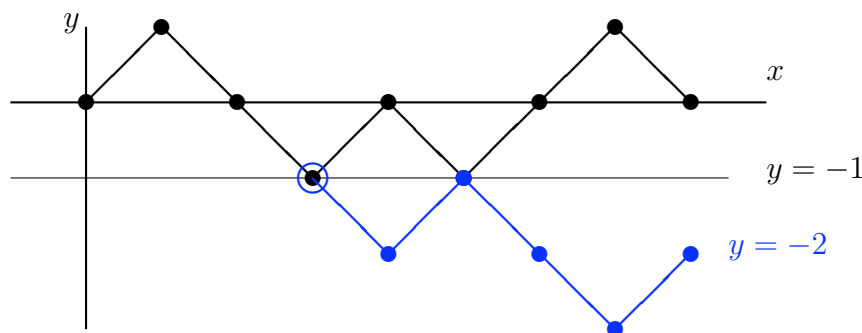
$$\binom{2n}{n}.$$

We want to calculate the number of “good” paths. But instead we count the number of “bad” paths since:

$$\text{total \# paths} = \text{\#good paths} + \text{\#bad paths}$$

$$\binom{2n}{n} = C(n) + (??)$$

The “bad” paths are those which, at some point go below the y -axis. So, they hit the horizontal line $y = -1$ at some point in time. The *reflection principle* says: Take the first point where the path hits the line $y = -1$ and reflect the rest of the path vertically through that line:



What happens is that the reflected path ends at $y = -2$. And conversely, any random walk which ends up at $y = -2$ starting at $y = 0$ must cross the line $y = -1$ at some point and we can reflect the blue path back up. This gives a bijection between the set of bad paths (paths which start and end at $y = 0$ and go to $y = -1$ at some point) and the set of all paths which start at $y = 0$ and end at $y = -2$ after $2n$ steps.

This second set has $\binom{2n}{n-1}$ elements since, out of $2n$ steps we need to take exactly $n-1$ steps to the left and $n+1$ steps to the right in order to arrive at 2 steps to the right at the end. So the number of good paths is

$$\begin{aligned} \# \text{ Dyck paths} &= \text{total } \# \text{ paths} - \# \text{ bad paths} \\ &= \binom{2n}{n} - \binom{2n}{n-1} \end{aligned}$$

Expand this out:

$$= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!}$$

Factor out common factors:

$$\begin{aligned} &= \frac{(2n)!}{(n-1)!n!} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\ &= \frac{(2n)!}{(n-1)!n!} \left[\frac{1}{n(n+1)} \right] \end{aligned}$$

Collect terms:

$$= \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \frac{(2n)!}{n!n!} = C(n).$$

□