4. Roots and reflections

4.1. roots of type A_{n-1}.

Definition 4.1. The *roots* of type A_{n-1} are the vectors in $\mathbb{R}^n$ of the form

$$e_i - e_j$$

First I had to explain to students what $\mathbb{R}^n$ means. The standard notation is:

$\mathbb{R}$ = the set of all real numbers.

$\mathbb{Z}$ = the set of all integers.

$\mathbb{N}$ = the set of all nonnegative integers. Unfortunately, some people think that 0 is not in this set. So, I will remind you each time I use this symbol.

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{R}\}$$

This represents n-dimensional space. Every point is specified by its n coordinates.

(1) How many roots are there?
(2) What is the length of each root?
(3) Which roots are perpendicular?
(4) What are the angles between the roots?
(5) What is the formula for the angle between two vectors?

4.1.1. *number of roots.* There are $n(n-1)$ roots α_{ij} since there are n choices for i and after that there are $n-1$ choices for j (since i, j must be distinct).

I forgot to say that the set of all roots is denoted Φ. I defined the *positive roots* to be the roots α_{ij} where $i > j$. The *simple roots* are the ones where i, j differ by 1. These are denoted

$$\alpha_i := \alpha_{i+1,i}, \ -\alpha_i = \alpha_{i,i+1}$$

4.1.2. *length.* OK, this is a dumb question. The roots all have length

$$\|X\| := \sqrt{\sum x_i^2} = \sqrt{2}.$$

4.1.3. *Which roots are $\perp$?* Two vectors are perpendicular if and only if their dot product:

$$X \cdot Y = \sum_{i=1}^n x_i y_i$$

is zero. But, the dot product of two vectors $\alpha_{ij}, \alpha_{k\ell}$ is zero if and only if the indices i, j, k, ℓ are distinct.

4.1.4. *formula for angle.* You take the second formula for the dot product

$$X \cdot Y = ||X|| \cdot ||Y|| \cos \theta$$

and solve for θ:

$$\theta = \cos^{-1} \frac{X \cdot Y}{||X|| \cdot ||Y||}$$

The dot product is $0, \pm 1, \pm 2$ and the denominator is always 2. So, the angles that you get are

$$\begin{array}{ll} \cos^{-1}(0) & = 90^\circ = \pi/2 \\ \cos^{-1}(1/2) & = 60^\circ = \pi/3 \\ \cos^{-1}(-1/2) & = 120^\circ = 2\pi/3 \\ \cos^{-1}(1) & = 0 \\ \cos^{-1}(-1) & = 180^\circ = \pi \end{array}$$

(1) $\theta = \pi/2$ $(= 90^\circ)$ or $\alpha \cdot \beta = 0$. This means the two roots α, β are perpendicular and this happens when i, j, k, ℓ are distinct.
(2) $\theta = \pi$. Then α, β are parallel and point in opposite directions. Since all roots have the same length the too roots are negatives of each other: α and $-\alpha$.
(3) $\theta = 0$. This means $\alpha = \beta$. (They have the same length and point in the same direction.)
(4) $\theta = \pi/3$. Then α, β form two sides of an equilateral triangle.
(5) $\theta = 2\pi/3$. Two consecutive (positive) simple roots α_i, α_{i+1} have this angle.

4.1.5. A_n *diagram.* The graph A_n consists of n vertices connected by $n - 1$ edges in a straight line. For example, A_4 stands for the following graph:

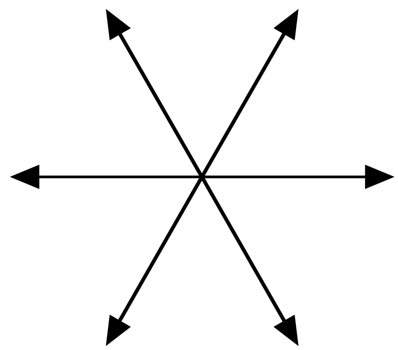
Each vertex represents a simple root. You draw a line between two roots if they are *not* perpendicular. When there is an edge connecting two roots, unless otherwise stated, the angle is $2\pi/3$.

The root system A_2 is drawn:

There are two simple roots forming an angle of 120°. There are also the negatives of these roots. On Wednesday we looked at the list of all roots in A_2:

$$(1, -1, 0),\ (-1, 1, 0),\ (0, 1, -1),\ (0, -1, 1),\ (1, 0, -1),\ (-1, 0, 1)$$

None of these roots are $\perp$. So the picture must be:



These are vectors in $\mathbb{R}^3$. But I only drew the plane given by:

$$x + y + z = 0$$

Recall that the equation of a plane in $\mathbb{R}^3$ is:

$$ax + by + z = d$$

where (a, b, c) is a vector which is perpendicular to the plane and d tells you how far away the plane is from the origin. In our case all roots have coordinates adding up to zero. So they lie in the $n-1$ dimensional hyperplane given by the equation

$$\sum x_i = 0$$

The perpendicular vector is $(1, 1, 1, \cdots, 1)$.

4.2. **reflections.** These are orthogonal transformations which fix a hyperplane. For example, switching x and y coordinates is a reflection through the line $x = y$ and "along" the vector $(1, -1)$. Reflection along the root $\alpha_{ij} = e_i - e_j$ switched the i-th and j-th coordinates.

Which reflections commute?

4.3. **other roots systems.** B_n, C_n, D_n, etc.

5. Catalan numbers and Narayama numbers

The *Catalan numbers* are given by

$$C(n) = \frac{1}{n}\binom{2n}{n-1}$$

It is the sum of *Narayama numbers* which are given by

$$N(k,n) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

Theorem 5.1. $C(n) = \sum_{k=1}^{n} N(k,n)$

There are lots of sets with a Catalan number of elements. Three of them that we will look at are the sets of binary trees, noncrossing partitions and clusters of type A_n.

5.1. **binary trees.** A binary tree is a rooted planar tree (one vertex is labelled as the root and the tree is embedded in the plane with root at the top) in which every node has two daughters, a left daughter and a right daughter.

5.2. **noncrossing partitions.** This is a partition of the set $\{1, 2\cdots, n\}$ into parts so that the parts don't cross when you draw the points in a circle and draw a convex curve surrounding each part.

5.3. **clusters of type A_n.** A complicated story.

6. Categories

6.1. **definition.** A *category* $\mathcal{C}$ is a collection $Ob(\mathcal{C})$ of *objects* A, B, C, etc. which are connected by *morphisms* $f : A \to B$, $g : B \to C$ which can be *composed* $g \circ f : A \to C$ so that composition is associative and each object has an identity $id_X : X \to X$.

6.2. **posets.** A *poset* is a set P with transitive, anti-reflexive relation $<$. The objects of P form a category if we think of the relation $a < b$ as a morphism $b \to a$. For example $n \to n-1 \to \cdots \to 2 \to 1$.

6.3. **$\mathbb{F}_2$-categories.** We will look at categories in which there is a zero morphism and at most one nonzero morphism between any two objects. $\mathbb{F}_2 = \{0, 1\}$ is the field with two elements. (Think $0 =$ *even*, $1 =$ *odd*. You can add and multiply: *odd+odd=even* or $1 + 1 = 0$.)

6.4. **cluster categories of type A_n over $\mathbb{F}_2$.** The objects are positive roots and negative projective roots $p_k := \alpha_{nk}$.

7. Research

When a new area of mathematics opens up, not all of the easy theorems have been discovered. So, students have a chance to find something new. I will show you how to do it.