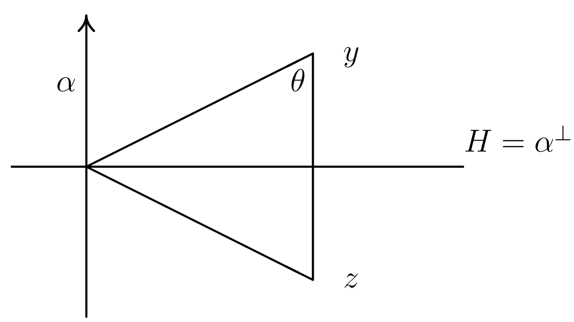
4.2. **reflections.** These are orthogonal transformations which fix a hyperplane. For example, switching x and y coordinates is a reflection through the line $x = y$ and "along" the vector $(1, -1)$. Reflection along the root $\alpha_{ij} = e_i - e_j$ switched the i-th and j-th coordinates.

Definition 4.2. Suppose that $\alpha = (a_1, a_2, \cdots, a_n)$ is any nonzero vector in $\mathbb{R}^n$. Then the perpendicular *hyperplane* is the set of all vectors x which are perpendicular to α. In other words $x \cdot \alpha = 0$, i.e.,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

We also discussed the equation for the reflection through this hyperplane.

If α is a vector (nonzero) in $\mathbb{R}^n$ and H is the perpendicular hyperplane, then the *reflection* "through H" (or "along α") is the mapping which sends $y \in \mathbb{R}^n$ to a point z which is equi-distant to H on the "other side."



By looking at the picture we decided that the formula must be:

$$z = \boxed{r_\alpha(y) = y - \frac{2y \cdot \alpha}{\alpha \cdot \alpha}\alpha}$$

I forgot to point out that, when $||\alpha|| = \sqrt{2}$ as it is in our case, this formula becomes really simple:

$$\boxed{r_\alpha(y) = y - (y \cdot \alpha)\alpha}$$

By looking at the picture for A_2 we see that the reflection of any root along any other root is another root. So, I gave the following definition of a (finite) root system:

Definition 4.3. A finite *root system* is defined a finite set of vectors in $\mathbb{R}^n$ (The set is called Φ, the vectors are called α, β, γ, etc.) so that the reflection of any $\alpha \in \Phi$ along $\beta \in \Phi$ is another element of Φ.

In the "crystallographic case" the root system is "simply laced" if all the roots have the same length. I will explain this later.

We decided that the following theorem was "obvious" and we didn't try to prove it. The correct statement of the theorem is:

Theorem 4.4. *In* $\mathbb{R}^2$*, a finite root system is given by taking any set of unit vectors so that the angle between any two vectors in the set is*

$$\frac{\pi k}{m}$$

where m*. In general, the lengths of the vectors need not be the same but the angles must be* $k\pi/m$*.*

I should have pointed out in class that, for any root α, $-\alpha$ must be a root since $-\alpha$ is the reflection of α along α.

The group generated by these reflections is the *dihedral group* D_m of order $2m$. This is the symmetry group of the regular m-gon. For example for A_2 it is $D_3 = S_3$, the symmetry group of an equilateral triangle.

Here is a proof:

Proof. Take the smallest angle between any two roots. Suppose that α, β are two roots which form this angle θ. Suppose that β is counterclockwise from α. Then $-\alpha$ is also a root and the reflection of $-\alpha$ along β is a root which is clockwise from β with an angle of θ, i.e., γ forms an angle of 2θ with α. If you then reflect $-\beta$ along γ you get a new root which is 3θ counterclockwise from α. Proceeding in this way, we will go all the way to a and there are two possibilities.

(1) Either we hit $-\alpha$ or
(2) we jump over $-\alpha$.

But the second case is not possible. If we jump over $-\alpha$ then the root $-\alpha$ will be in the middle of an an angle θ. So, we would get a smaller angle.

So, it has to be Case 1. We have to hit $-\alpha$ in a finite number of steps, say m steps. Then $\theta = \pi/m$. □

We divide the set of roots into "positive" and "negative" roots:

$$\Phi = \Phi_+ \coprod \Phi_-$$

The way to do this is to choose a hyperplane which does not contain any of the roots. Then the roots on one side of the hyperplane are called positive and the ones on the other side are called negative. This is random but any two choices can be shown to be equivalent.

Definition 4.5. The set of *simple roots* $\alpha_1, \alpha_2, \cdot, \alpha_n$ is defined to be a set of positive roots with two properties:

(1) The simple roots form a basis for the subspace spanned by all the roots. In other words every root is a linear combination of simple roots.
(2) Every positive root is a nonnegative linear combination of simple roots, i.e.,
$$\beta = \sum b_i \alpha_i$$
where $\beta_i \geq 0$.

These two properties determine the set of simple roots because there is only one set with this property. In the case of the 2-dimensional case, the simple roots are the two which are closest to the separating hyperplane.