4. Roots and reflections

4.1. roots of type A_{n-1}.

Definition 4.1. The *roots* of type A_{n-1} are the vectors in $\mathbb{R}^n$ of the form

$$e_i - e_j$$

First I had to explain to students what $\mathbb{R}^n$ means. The standard notation is:
$\mathbb{R}$ = the set of all real numbers.
$\mathbb{Z}$ = the set of all integers.
$\mathbb{N}$ = the set of all nonnegative integers. Unfortunately, some people think that 0 is not in this set. So, I will remind you each time I use this symbol.

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \,|\, x_i \in \mathbb{R}\}$$

This represents n-dimensional space. Every point is specified by its n coordinates.

(1) How many roots are there?
(2) What is the length of each root?
(3) Which roots are perpendicular?
(4) What are the angles between the roots?
(5) What is the formula for the angle between two vectors?

4.1.1. *number of roots.* There are $n(n-1)$ roots α_{ij} since there are n choices for i and after that there are $n-1$ choices for j (since i, j must be distinct).

I forgot to say that the set of all roots is denoted Φ. I defined the *positive roots* to be the roots α_{ij} where $i > j$. The *simple roots* are the ones where i, j differ by 1. These are denoted

$$\alpha_i := \alpha_{i+1,i},\ -\alpha_i = \alpha_{i,i+1}$$

4.1.2. *length.* OK, this is a dumb question. The roots all have length

$$\|X\| := \sqrt{\sum x_i^2} = \sqrt{2}.$$

4.1.3. *Which roots are $\perp$?* Two vectors are perpendicular if and only if their dot product:

$$X \cdot Y = \sum_{i=1}^{n} x_i y_i$$

is zero. But, the dot product of two vectors $\alpha_{ij}, \alpha_{k\ell}$ is zero if and only if the indices i, j, k, ℓ are distinct.

4.1.4. *formula for angle.* You take the second formula for the dot product

$$X \cdot Y = ||X|| \cdot ||Y|| \cos\theta$$

and solve for θ:

$$\theta = \cos^{-1} \frac{X \cdot Y}{||X|| \cdot ||Y||}$$

The dot product is $0, \pm 1, \pm 2$ and the denominator is always 2. So, the angles that you get are

$$\begin{array}{ll} \cos^{-1}(0) & = 90^\circ = \pi/2 \\ \cos^{-1}(1/2) & = 60^\circ = \pi/3 \\ \cos^{-1}(-1/2) & = 120^\circ = 2\pi/3 \\ \cos^{-1}(1) & = 0 \\ \cos^{-1}(-1) & = 180^\circ = \pi \end{array}$$

(1) $\theta = \pi/2$ $(= 90^\circ)$ or $\alpha \cdot \beta = 0$. This means the two roots α, β are perpendicular and this happens when i, j, k, ℓ are distinct.
(2) $\theta = \pi$. Then α, β are parallel and point in opposite directions. Since all roots have the same length the too roots are negatives of each other: α and $-\alpha$.
(3) $\theta = 0$. This means $\alpha = \beta$. (They have the same length and point in the same direction.)
(4) $\theta = \pi/3$. Then α, β form two sides of an equilateral triangle.
(5) $\theta = 2\pi/3$. Two consecutive (positive) simple roots α_i, α_{i+1} have this angle.

4.1.5. A_n *diagram.* The graph A_n consists of n vertices connected by $n-1$ edges in a straight line. For example, A_4 stands for the following graph:

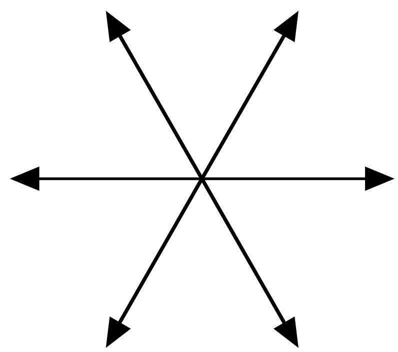
Each vertex represents a simple root. You draw a line between two roots if they are *not* perpendicular. When there is an edge connecting two roots, unless otherwise stated, the angle is $2\pi/3$.

The root system A_2 is drawn:

There are two simple roots forming an angle of 120°. There are also the negatives of these roots. On Wednesday we looked at the list of all roots in A_2:

$$(1, -1, 0),\ (-1, 1, 0),\ (0, 1, -1),\ (0, -1, 1),\ (1, 0, -1),\ (-1, 0, 1)$$

None of these roots are $\perp$. So the picture must be:



These are vectors in $\mathbb{R}^3$. But I only drew the plane given by:

$$x + y + z = 0$$

Recall that the equation of a plane in $\mathbb{R}^3$ is:

$$ax + by + z = d$$

where (a, b, c) is a vector which is perpendicular to the plane and d tells you how far away the plane is from the origin. In our case all roots have coordinates adding up to zero. So they lie in the $n-1$ dimensional hyperplane given by the equation

$$\sum x_i = 0$$

The perpendicular vector is $(1, 1, 1, \cdots, 1)$.

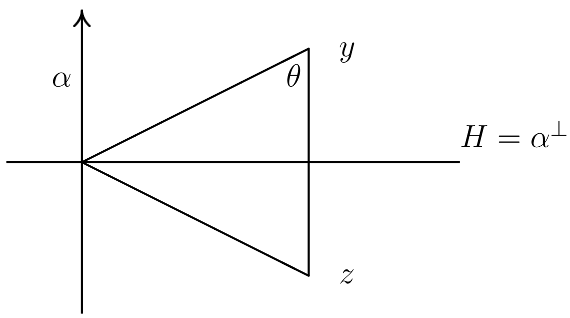
4.2. **reflections.** These are orthogonal transformations which fix a hyperplane. For example, switching x and y coordinates is a reflection through the line $x = y$ and "along" the vector $(1, -1)$. Reflection along the root $\alpha_{ij} = e_i - e_j$ switched the i-th and j-th coordinates.

Definition 4.2. Suppose that $\alpha = (a_1, a_2, \cdots, a_n)$ is any nonzero vector in $\mathbb{R}^n$. Then the perpendicular *hyperplane* is the set of all vectors x which are perpendicular to α. In other words $x \cdot \alpha = 0$, i.e.,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

We also discussed the equation for the reflection through this hyperplane.

If α is a vector (nonzero) in $\mathbb{R}^n$ and H is the perpendicular hyperplane, then the *reflection* "through H" (or "along α") is the mapping which sends $y \in \mathbb{R}^n$ to a point z which is equi-distant to H on the "other side."



By looking at the picture we decided that the formula must be:

$$z = \boxed{r_\alpha(y) = y - \frac{2y \cdot \alpha}{\alpha \cdot \alpha}\alpha}$$

I forgot to point out that, when $||\alpha|| = \sqrt{2}$ as it is in our case, this formula becomes really simple:

$$\boxed{r_\alpha(y) = y - (y \cdot \alpha)\alpha}$$

By looking at the picture for A_2 we see that the reflection of any root along any other root is another root. So, I gave the following definition of a (finite) root system:

Definition 4.3. A finite *root system* is defined a finite set of vectors in $\mathbb{R}^n$ (The set is called Φ, the vectors are called α, β, γ, etc.) so that the reflection of any $\alpha \in \Phi$ along $\beta \in \Phi$ is another element of Φ.

In the "crystallographic case" the root system is "simply laced" if all the roots have the same length. I will explain this later.

We decided that the following theorem was "obvious" and we didn't try to prove it. The correct statement of the theorem is:

Theorem 4.4. *In $\mathbb{R}^2$, a finite root system is given by taking any set of unit vectors so that the angle between any two vectors in the set is*

$$\frac{\pi k}{m}$$

where m. In general, the lengths of the vectors need not be the same but the angles must be $k\pi/m$.

I should have pointed out in class that, for any root α, $-\alpha$ must be a root since $-\alpha$ is the reflection of α along α.

The group generated by these reflections is the *dihedral group* D_m of order $2m$. This is the symmetry group of the regular m-gon. For example for A_2 it is $D_3 = S_3$, the symmetry group of an equilateral triangle.

Here is a proof:

Proof. Take the smallest angle between any two roots. Suppose that α, β are two roots which form this angle θ. Suppose that β is counterclockwise from α. Then $-\alpha$ is also a root and the reflection of $-\alpha$ along β is a root which is clockwise from β with an angle of θ, i.e., γ forms an angle of 2θ with α. If you then reflect $-\beta$ along γ you get a new root which is 3θ counterclockwise from α. Proceeding in this way, we will go all the way to a and there are two possibilities.

(1) Either we hit $-\alpha$ or
(2) we jump over $-\alpha$.

But the second case is not possible. If we jump over $-\alpha$ then the root $-\alpha$ will be in the middle of an an angle θ. So, we would get a smaller angle.

So, it has to be Case 1. We have to hit $-\alpha$ in a finite number of steps, say m steps. Then $\theta = \pi/m$. □

We divide the set of roots into "positive" and "negative" roots:

$$\Phi = \Phi_+ \coprod \Phi_-$$

The way to do this is to choose a hyperplane which does not contain any of the roots. Then the roots on one side of the hyperplane are called positive and the ones on the other side are called negative. This is random but any two choices can be shown to be equivalent.

Definition 4.5. The set of *simple roots* $\alpha_1, \alpha_2, \cdot, \alpha_n$ is defined to be a set of positive roots with two properties:

(1) The simple roots form a basis for the subspace spanned by all the roots. In other words every root is a linear combination of simple roots.
(2) Every positive root is a nonnegative linear combination of simple roots, i.e.,
$$\beta = \sum b_i \alpha_i$$
where $\beta_i \geq 0$.

These two properties determine the set of simple roots because there is only one set with this property. In the case of the 2-dimensional case, the simple roots are the two which are closest to the separating hyperplane.

4.3. **other roots systems.** B_n, C_n, D_n, etc.

Definition 4.6. A root system is called *crystallographic* if the coefficient in the reflection formula is always an integer:

$$\frac{2\alpha \cdot \beta}{\alpha \cdot \alpha} \in \mathbb{Z}$$

This number is denoted $\langle \beta, \alpha \rangle$.

In other words,

$$r_\alpha(\beta) = \beta - n\alpha, \quad \text{for some integer } n$$

The sign can also be positive since

$$\beta = r_\alpha(\beta) + n\alpha.$$

(When you reflect the root back the coefficient has the opposite sign.)

We found that there are four possibilities:

(1) $(n = 0)$ α, β are perpendicular in this case. This is called $A_1 \times A_1$.
(2) $(n = 1)$ α, β form an angle of $60^\circ = \pi/3$. This is A_2 which we discussed earlier.
(3) $(n = 2)$ The angle is $45^\circ = \pi/4$. This is called B_2.
(4) $(n = 3)$ The angle is $30^\circ = \pi/6$. This is G_2.

We also proved that $n = 4$ is not possible. And the proof works the same way for $n \geq 4$.

Theorem 4.7. *In a crystallographic root system, the coefficient* $n = \langle \beta, \alpha \rangle = 2\alpha \cdot \beta / \alpha \cdot \alpha$ *is* $n = 0, \pm 1, \pm 2$ *or* ± 3.

Proof. This follows from the triangle inequality and the pythagorean theorem. The triangle inequality says that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides:

$$c \leq a + b.$$

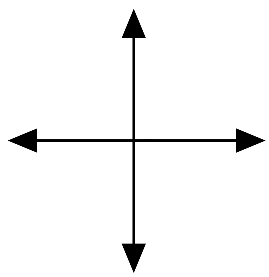
Suppose that $\rho_\alpha(\beta) = \beta - n\alpha$ where α is a unit vector and $n \geq 4$. We can draw α on the x-axis by rotating the picture if necessary. So, $\alpha = (1, 0)$. Then r_α is reflection through the y-axis. So, the line from β to $r_\alpha(\beta)$ is horizontal. We get a right triangle: If $\beta = (x, y)$ then, we don't know what y is but we know that $x = n/2$ which is 2 or more. This means that the hypotenuse is larger:

$$||\beta|| > 2.$$

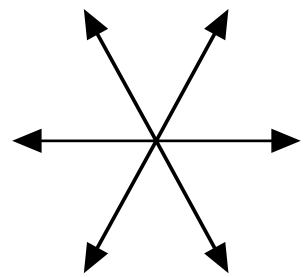
Next we looked at $r_\beta(\alpha)$.

$$r_\beta(\alpha) = \alpha - m\beta$$

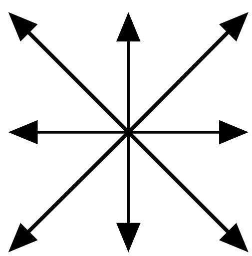
If the root system is crystallographic, then m must be an integer. So, it is at least 1. But α and $r_\beta(\alpha)$ are both unit vectors. They form two sides of a triangle and the third side is $m\beta$ which has length greater than 2. This is impossible. So the number $n = 4, 5, 6$, etc are not possible. The examples show that $n = 0, 1, 2, 3$ are possible. $\square$



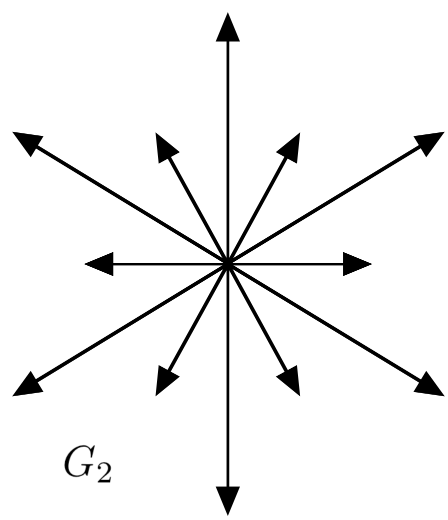
$A_1 \times A_1$



A_2



B_2



G_2