5.5. **noncrossing partitions.** This is a partition of the set $\{1, 2 \cdots, n\}$ into parts so that the parts don't cross when you draw the points in a circle and draw a convex curve surrounding each part.

Here is the plan of how to study any new concept:

1. Write the definition.
2. Do examples. Look for patterns.
3. Ask questions.
   1. (a) What is known?
   2. (b) Problems
   3. (c) What is it? What does it mean?
4. The real point is to work with the concept until you feel comfortable with it.

5.5.1. *definition.*

Definition 5.19. A *partition* of a set S is a way of writing the set as a disjoint union of subsets:

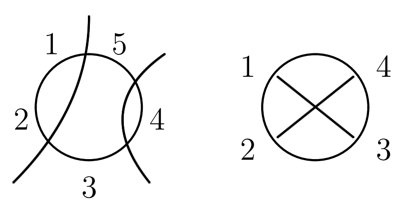
$$S = A \coprod B \coprod C \coprod \cdots$$

Here are two example of partitions of the set $[n] := \{1, 2, 3, \cdots, n\}$.

$$[5] = \{1, 2\} \coprod \{3, 5\} \coprod \{4\}$$

$$[4] = \{1, 3\} \coprod \{2, 4\}$$

The first partition is said to be "noncrossing" and the second is a "crossing" partition. The reason is that, when you draw the points in order in a circle and connect numbers in the same part to each other with straight lines (or convex ovals) then the lines cross in the second case but not in the first:



Definition 5.20. A partition of n (i.e. of $[n]$) is called *noncrossing* if there do not exist integers $1 \leq a < b < c < d \leq n$ so that a, c lie in one part and b, d lie in another part.

It is known that there are a Catalan number of noncrossing partitions.

5.5.2. $n = 2$. We started with the simplest case: $n = 2$. Then there are two noncrossing partitions:

$$(1)(2) \quad \text{and} \quad (12)$$

Students observed that when a part has only one element, it cannot be crossing.

5.5.3. $n = 3$. In this case there are five noncrossing partitions. Influenced by the analysis of the $n = 2$ case, students quickly thought of the partition in which each element is separate and the one where there is only one part:

$$(1)(2)(3) \quad \text{and} \quad (123)$$

Later they found the three others:

$$(1)(23), \quad (12)(3) \quad \text{and} \quad (2)(31)$$

I drew the last one in a circle to show that it is noncrossing. I asked for patterns. The pattern that some students thought described the 3 new cases was:

$$\binom{n}{1} = \binom{3}{1} = 3.$$

5.5.4. $n = 4$. I pointed out that $C(2) = 2$ and $C(3) = 5$. So, there should be $C(3) = 14$ noncrossing partitions of 3.

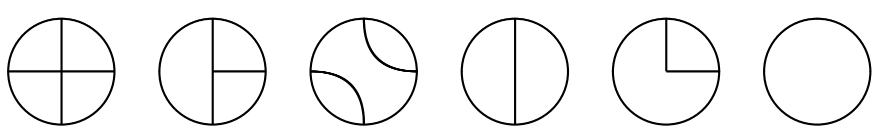
Students quickly found the 6 which were analogous to the $n = 2$ and $n = 3$ cases:

$$(1)(2)(3)(4) \quad \text{and} \quad (1234)$$
$$(1)(234), \quad (2)(341), \quad (3)(412) \quad \text{and} \quad (123)(4)$$

Then we had to find the other 8. I don't remember in which order we found them. I will sort them by shape here:

$$(12)(34), \quad (23)(41)$$
$$(12)(3)(4), \quad (23)(4)(1), \quad (34)(1)(2), \quad (41)(2)(3)$$
$$(13)(2)(4), \quad (24)(1)(3)$$

The shapes that we found were:



1 4 2 2 4 1

What kind of patterns will there be for $n = 4$?