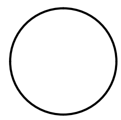
5.5.5. $n = 5$. Today we looked at noncrossing partitions of $n = 5$. We expect to get:

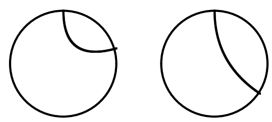
$$C(5) = 42$$

noncrossing partitions. We got 10 patterns:

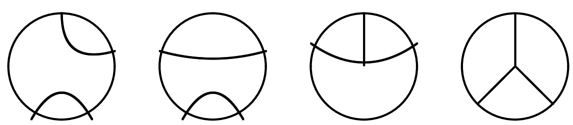
(1) 1 with one part



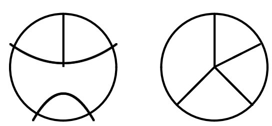
(2) 2 patterns with 2 parts. Each gave 5 ncp's for $2 \times 5 = 10$ ncp's:



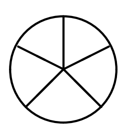
(3) 4 patterns with 3 parts. Each gives 5 ncp's for $4 \times 5 = 20$ npc's:



(4) 2 patterns with 4 parts each gives 5 npcs for a total of 10 npc's:



(5) 1 pattern with 5 part.



The class, in particular Liz, discovered the following pattern:

Theorem 5.21. *If p is a prime number then $C(p) - 2$ is divisible by p.*

Proof. The geometric proof is that, except for the two extreme cases of one part and p parts, every pattern will give p ncp's by rotating the shape.

An algebraic proof, which I tried but failed to do in class is the following: Each of the Narayana numbers $N(p, k)$ is divisible by p

except for the first ($k = 1$) and last ($k = p$):

$$N(p, k) = \frac{1}{p}\binom{p}{k}\binom{p}{k-1}$$

When $1 < k < p$, both binomials are divisible by p since all the numbers in the denominators are less than p. After dividing by p, the number $N(p, k)$ is still divisible by p. The first and last Narayana numbers are always equal to 1:

$$N(n, 1) = \frac{1}{n}\binom{n}{1}\binom{n}{0} = \frac{n}{n} = 1.$$

$$N(n, n) = \frac{1}{n}\binom{n}{n}\binom{n}{n-1} = \frac{n}{n} = 1.$$

□