7. Categories

The "categorification" of cluster combinatorics began in the 21st century. Whereas the first results about clusters were published by Fuss in 1791. We will look at this new idea and try to understand it.

7.1. **definition.** A *category* $\mathcal{C}$ is a collection $Ob(\mathcal{C})$ of *objects* A, B, C, etc. which are connected by *morphisms* $f : A \to B$, $g : B \to C$ which can be *composed* $g \circ f : A \to C$ so that composition is associative and each object has an identity $id_X : X \to X$.

The classical example of a category is the category $\mathcal{S}ets$ of all sets and mappings. Here $Ob(\mathcal{S}ets)$ is the collection of all sets. Thus entire sets, such as $[n], \mathbb{Z}, \mathbb{R}$ are elements of $Ob(\mathcal{S}ets)$, i.e. points in the category. These points are connected by arrows:

$$\mathrm{Hom}_{\mathcal{S}ets}(X, Y) = \{f : X \to Y\}$$

In class we did the case $X = [2], Y = [3]$ then

$$[3]^{[2]} = Y^X := \mathrm{Hom}_{\mathcal{S}ets}([2], [3])$$

has $3^2 = 9$ elements. So, the category $\mathcal{S}ets$ has 9 arrows going from [2] to [3]. There are an infinite number of arrows from $\mathbb{Z}$ to $\mathbb{R}$.

I went through this definition in much more detail than perhaps the students wanted to hear:

Definition 7.1. A *category* $\mathcal{C}$ consists of

(1) $Ob(\mathcal{C})$, the collection of objects: $X, Y, Z, \cdots$

(2) For every pair of objects X, Y we have a set of morphisms

$$\mathrm{Hom}_{cC}(X, Y); \{f : X \to Y\}$$

(3) You can compose arrows (morphisms)

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \Rightarrow \quad g \circ f : X \to Z$$

In words: Given any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{cC}(Y, Z)$ we have $g \circ f \in \mathrm{Hom}_{cC}(X, Z)$. And composition is *associative*:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

(4) Every object X has an *identity morphism*

$$id_X : X \to X$$

which satisfies the property: $f \circ id_X = f$ for all arrows $f : X \to Y$ and $id_X \circ g = g$ for all arrows $g : Z \to X$.

Example 7.2. Suppose that G is any group. Then there is a category which we called $\mathcal{G}$ with one object $*$ and one morphism $g : * \to *$ for every element $g \in G$. The composition of morphisms is defined to be group multiplication:

$$* \xrightarrow{h} * \xrightarrow{g} * \quad \text{gives} \quad g \circ h = gh \quad \text{by definition.}$$

Then composition is associative:

$$(f \circ g) \circ h = (fg)h = f(gh) = f \circ (g \circ h)$$

The identity arrow is given by the identity of the group:

$$id_* = e$$

and we checked that it is the identity:

$$id_* \circ f = e \circ f = ef = f = f \circ id_*$$

Then we did an example of the example: Suppose $G = \langle \mathbb{Z}/2, + \rangle$. Then we get a category $\mathcal{C}_+$ with one object $*$ and two morphisms $0, 1 : * \to *$ with composition law given by the chart:

$+$	0	1
0	0	1
1	1	0

Since this is an specific example of the previous example, it is a category. The identity of $*$ is $id_* = 0$.

Finally, we started the example of the multiplication rule for composition: The category $\mathcal{C}_\times$ has one object $*$, two morphisms $0, 1 : * \to *$ with composition given by the chart:

$\times$	0	1
0	0	0
1	0	1

The identity of $*$ in this category is $id_* = 1$.

Question 7.3. *Describe all categories with one object $*$ and two morphisms $a, b : * \to *$. In particular, are they all isomorphic to the above two examples?*