Example 7.2. Suppose that G is any group. Then there is a category which we called $\mathcal{G}$ with one object $*$ and one morphism $g : * \to *$ for every element $g \in G$. The composition of morphisms is defined to be group multiplication:

$$* \xrightarrow{h} * \xrightarrow{g} * \quad \text{gives} \quad g \circ h = gh \quad \text{by definition.}$$

Then composition is associative:

$$(f \circ g) \circ h = (fg)h = f(gh) = f \circ (g \circ h)$$

The identity arrow is given by the identity of the group:

$$id_* = e$$

and we checked that it is the identity:

$$id_* \circ f = e \circ f = ef = f = f \circ id_*$$

Then we did an example of the example: Suppose $G = \langle \mathbb{Z}/2, + \rangle$. Then we get a category $\mathcal{C}_+$ with one object $*$ and two morphisms $0, 1 : * \to *$ with composition law given by the chart:

+	0	1
0	0	1
1	1	0

Since this is an specific example of the previous example, it is a category. The identity of $*$ is $id_* = 0$.

Finally, we started the example of the multiplication rule for composition: The category $\mathcal{C}_\times$ has one object $*$, two morphisms $0, 1 : * \to *$ with composition given by the chart:

×	0	1
0	0	0
1	0	1

The identity of $*$ in this category is $id_* = 1$.

Question 7.3. *Describe all categories with one object $*$ and two morphisms $a, b : * \to *$. In particular, are they all isomorphic to the above two examples?*

7.1.1. *examples.* On Monday we did this example. Suppose you have a category with two objects: A, B and four arrows: $A \to A, A \to B, B \to A, B \to B$. The question is: What are these arrows? What should they be called? What are their compositions?

The answer is

(1) Each object has an identity. So the two loops must be:
$$id_A : A \to A, \quad id_B : B \to B.$$
(2) Call the other two arrows $f : A \to B, g : B \to A$.
(3) The compositions must be the identities:
$$f \circ g = id_B, \quad g \circ f = id_A$$
the reason is that they cannot be anything else:
$$g \circ f : A \xrightarrow{f} B \xrightarrow{g} A$$
Since $g \circ f$ is an arrow from A to A it must be id_A.

The other example I started but did not finish was:

Suppose you have a category with two objects: A, B and five arrows: $A \to A, A \to B, B \to A$ and two loops at B: $B \to B$. The question is: What are these arrows? What should they be called? What are their compositions?

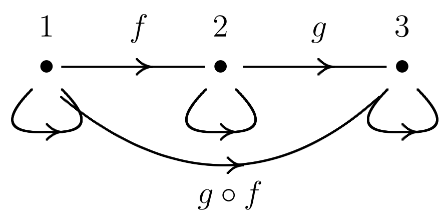
(1) We have the identities id_A, id_B
(2) We have two arrows $f : A \to B, g : B \to A$.
(3) We have one more arrow $h : B \to B$.
(4) The composition $g \circ f : A \to A$ must be the identity of A since it can't be anything else.
(5) The composition $f \circ g : B \to B$ is either id_B or $f \circ g = h$.
(6) This implies that $h \circ h = h$:
$$h \circ h = (f \circ g) \circ (f \circ g) = f \circ id_A \circ g = f \circ g = h.$$
(7) All other compositions are uniquely determined.

7.2. **posets.** A *poset* is a set P with transitive, anti-reflexive relation $<$. In other words, $a < b < c$ implies $a < c$ and $a < b$ implies $b \not\leq a$. We have to take the $\leq$ relation to get a category.

7.2.1. *integers with usual ordering.* I started with an infinite category $\mathcal{Z}$. This has objects all integers. I.e., each integer is one object. There is a unique morphism $a \to b$ if $a \leq b$. If $a > b$ there is no arrow $a \to b$:

(1) $Ob(\mathcal{Z}) = \mathbb{Z}$, i.e., the objects are $\cdots - 2, -1, 0, 1, 2, \cdots$
(2) $Ar(\mathcal{Z})$: $a \to b$ if and only if $a \leq b$.

To get a finite category, take only a finite set of integers. For example take just 1,2,3. This makes a category with three objects and 6 morphisms:



This drawing has redundant information which I want to suppress. First, each object has an identity loop. So, we don't draw them. We know they are always there and we know what their names are (id_1, id_2, id_3). Also, the composition of the two basic arrows f, g is necessarily present by definition of a category. We also know its name $(g \circ f)$. So, we don't draw it. What I draw is:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

Some people like to draw objects as points. Other people like to write the names of the points:

$$1 \longrightarrow 2 \longrightarrow 3$$

7.2.2. *Hasse diagram.* A *Hasse diagram* is a graph drawn in a plane where all the edges are going up-down. There are no horizontal edges. When there is an edge connecting two points a, b with a below and b above this means $a < b$. By transitivity $a < b < c$ gives $a < c$. In other words, $x < y$ if there is a path from x to y which is always going up.