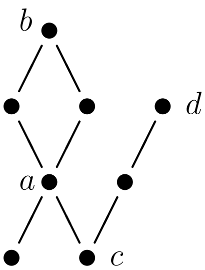
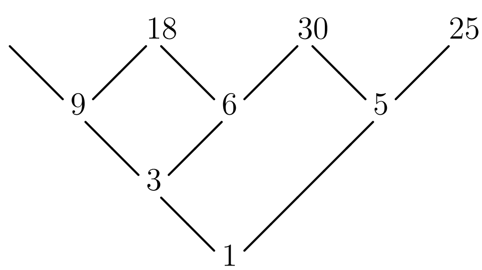



For example, in this Hasse drawing, $a \leq b, c \leq a$ and $c \leq d$ but $a, d$ are not comparable.

Other examples are given by division:

Take positive integers as objects and take one arrow $a \to b$ if $a$ divides $b$. This gives a Hasse diagram like this:




Tomorrow we will add zeros. This will allow us to make loops without backward nonzero arrows.

7.3. **Additive categories.** Today I talked about additive categories. I gave the general definition then some particular examples. I explained the dotted line notation and we did some calculations.

7.3.1. *general definition.* An *additive category* is a category with two properties.

(1) Hom sets are additive groups. For any two objects $A, B$ in an additive category $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \{f : A \to B\}$$

is an additive group. This means it is an abelian group with operation written as addition (instead of the usual multiplication notation for groups). In other words, given two morphisms (arrows, homomorphisms) $f, g : A \to B$ you have their sum which is another morphism $f + g : A \to B$. You also have the *zero morphism* $0 : A \to B$ which is the identity of the group. This means that

$$0 + f = f = f + 0.$$

There is also a negative $-f$ so that

$$f + (-f) = 0 = (-f) + f.$$

(2) Composition is *biadditive*. This is also called the *distributive* law. Give $f, k : A \to B$ and $g, h : B \to C$ we have:

$$(g + h) \circ f = (g \circ f) + (h \circ f)$$

$$h \circ (f + k) = h \circ f + h \circ k$$

I pointed out that composition with 0 is 0:

**Proposition 7.4.** *In an additive category, composition of 0 with any morphism gives 0.*

*Proof.* Since $0 + 0 = 0$,

$$0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$$

By the cancellation law (for groups), this gives $0 \circ f = 0$. Similarly, $f \circ 0 = 0$. □

Writing these notes, I realized that some of you may not know the cancellation rule for groups:

**Lemma 7.5.** *Cancellation holds in any group. I.e., $ax = ay$ implies $x = y$ and $ax = a$ implies $x = e$. Similarly, $xb = yb \Rightarrow x = y$.*

*Proof.* Multiply on the left with $a^{-1}$:

$$\underbrace{a^{-1}a}_{e}x = \underbrace{a^{-1}a}_{e}y$$

So, $x = ex = ey = y$. When there is no $y$ you get $x = ex = e$. $\square$

7.3.2. *categories over* $\mathbb{F}_2$. I made this fancy definition really simple by taking cluster categories (of type $A_n$) over $\mathbb{F}_2$. This is the field with 2 elements $\mathbb{F}_2 = \{0, 1\}$. The hom sets are really simple. $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has either 1 or 2 elements:

$$\mathrm{Hom}_{\mathcal{C}}(A,B) = \begin{cases} \{0\} & \text{or} \\ \{0, f\} & \text{0 and a nonzero } f \end{cases}$$

Notice that:

(1) There is always a zero arrow: $0 : A \to B$.
(2) There might be a nonzero arrow $f : A \to B$.

Addition is given by $0+0 = 0$, $0+f = f = f+0$ and $f+f = 0$. That is because $-f = f$ (since it can't be anything else).

Given that composition is biadditive (distributive), what are the possible compositions? Well there are only four possibilities for two arrows being composed:

$$A \xrightarrow{0} B \xrightarrow{0} C \quad \text{gives} \quad A \xrightarrow{0} C$$

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \text{gives} \quad A \xrightarrow{0 \text{ or } h} C$$

$$A \xrightarrow{f} B \xrightarrow{0} C \quad \text{gives} \quad A \xrightarrow{0} C$$

$$A \xrightarrow{0} B \xrightarrow{g} C \quad \text{gives} \quad A \xrightarrow{0} C$$

We know that composition with 0 gives zero. But composition of two nonzero morphisms $f : A \to B$ and $g : B \to C$ could give either 0 or a nonzero morphism $h = g \circ f : A \to C$. It depends.

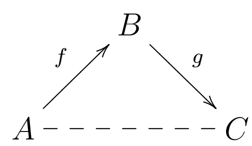
7.3.3. *cycle example.* I returned to the cycle example:

$$\begin{array}{ccc} B & \xleftarrow{f} & A \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle k} \\ C & \xrightarrow[h]{} & D \end{array}$$

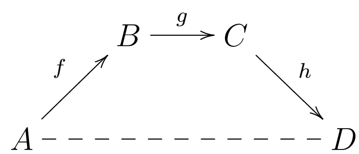
I gave two possible composition laws.

(1) Let $g \circ f = 0, h \circ g = 0$, etc (all compositions of two nonidentity morphism are zero).
(2) Let $h \circ g \circ f = 0, k \circ h \circ g = 0$, etc., i.e. all compositions of three nonidentity morphisms are zero. (And $g \circ f \neq 0$)

7.3.4. *dotted line notation.* People use dotted lines to indicate which compositions are zero. For example:

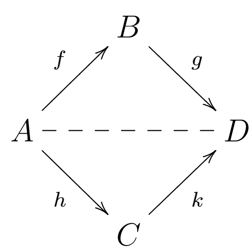



means $g \circ f = 0$.



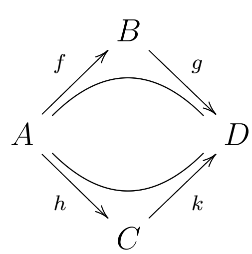

means $h \circ g \circ f = 0$.

The dotted line means that there is an equation among the paths going from one point to another. For example:



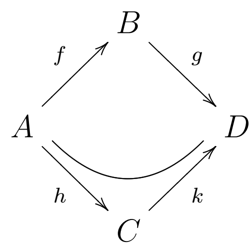

means $g \circ f = k \circ h$.

If we wanted to indicate that compositions are zero we would draw two dotted lines. (The curved lines are supposed to be dotted lines. I couldn't figure out how to type it.)



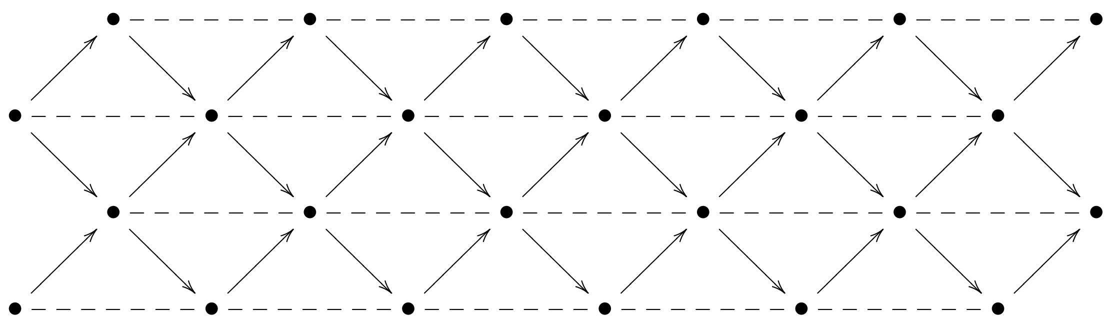

This means $g \circ f = 0$ and $k \circ h = 0$.

In other words, each dotted line indicates one equation. If you want only one zero relation you draw only one dotted curved line:

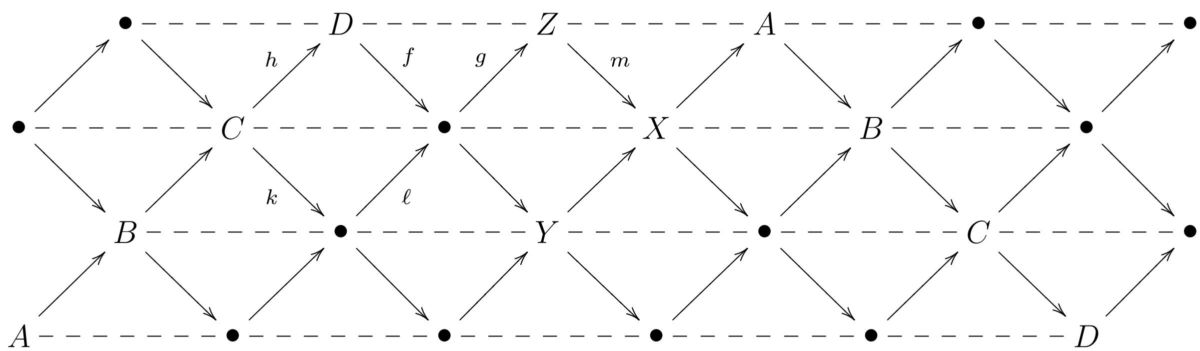



This means $k \circ h = 0$.

7.4. **cluster categories of type $A_n$ over $\mathbb{F}_2$.** Now that we know the notation, I can write down the cluster category. It is a repeating pattern:



This is actually on a Möbius strip. The objects $A, B, C, D$ on the left are the same as those on the right. This is the cluster category of type $A_4$.




Going step-by-step, I asked students to calculate the following. More precisely, the question is: Are these groups zero or nonzero?

(1) $\mathrm{Hom}_{\mathcal{C}}(D, Z) = ?$
(2) $\mathrm{Hom}_{\mathcal{C}}(C, Z) = ?$
(3) $\mathrm{Hom}_{\mathcal{C}}(C, X) = ?$

(4) $\text{Hom}_{\mathcal{C}}(C, Y) =?$

First, students realized that $\text{Hom}_{\mathcal{C}}(D, Z) = 0$ since the only path from $D$ to $Z$ is the composition $g \circ f$ which is equal to zero because of the dotted line from $D$ to $Z$.

Next, we looked at $\text{Hom}_{\mathcal{C}}(C, Z)$. There are two paths from $C$ to $Z$. The first is zero since $g \circ f = 0$ by the previous calculation.

$$\underbrace{g \circ f}_{0} \circ h = 0$$

The other path is

$$g \circ \ell \circ k = g \circ f \circ h = 0$$

which is equal to the previous path and is thus zero by the *mesh relation*

$$f \circ h = \ell \circ k$$

Students got the hang of this and they realized that $\text{Hom}_{\mathcal{C}}(C, X) = 0$ since all paths from $C$ to $X$ are equivalent, by repeated use of mesh relations, to the path

$$m \circ g \circ f \circ h = 0$$

Finally, we came to $\text{Hom}_{\mathcal{C}}(C, Y)$. This is nonzero. There are three path from $C$ to $Y$ and they are all equal. But none of these goes through a zero relation. So, they are not necessarily zero. The rule about these diagrams is that all of the information is in the diagram. If a composition of paths is zero, it must be as a consequence of the information in the drawing. This not being the case, the three paths are not zero.

Students asked: Where are the clusters?

It seems that I need one more lecture to show where the clusters enter in this category. This will tie everything together and we can go on to discuss questions and problems.