

(4) $\text{Hom}_{\mathcal{C}}(C, Y) = ?$

First, students realized that $\text{Hom}_{\mathcal{C}}(D, Z) = 0$ since the only path from D to Z is the composition $g \circ f$ which is equal to zero because of the dotted line from D to Z .

Next, we looked at $\text{Hom}_{\mathcal{C}}(C, Z)$. There are two paths from C to Z . The first is zero since $g \circ f = 0$ by the previous calculation.

$$\underbrace{g \circ f}_{0} \circ h = 0$$

The other path is

$$g \circ \ell \circ k = g \circ f \circ h = 0$$

which is equal to the previous path and is thus zero by the *mesh relation*

$$f \circ h = \ell \circ k$$

Students got the hang of this and they realized that $\text{Hom}_{\mathcal{C}}(C, X) = 0$ since all paths from C to X are equivalent, by repeated use of mesh relations, to the path

$$m \circ g \circ f \circ h = 0$$

Finally, we came to $\text{Hom}_{\mathcal{C}}(C, Y)$. This is nonzero. There are three paths from C to Y and they are all equal. But none of these goes through a zero relation. So, they are not necessarily zero. The rule about these diagrams is that all of the information is in the diagram. If a composition of paths is zero, it must be as a consequence of the information in the drawing. This not being the case, the three paths are not zero.

Students asked: Where are the clusters?

It seems that I need one more lecture to show where the clusters enter in this category. This will tie everything together and we can go on to discuss questions and problems.

7.5. Clusters in the cluster category. In the cluster category of type A_n , the clusters are defined to be sets of n “indecomposable” objects which do not “extend” each other. I will explain the (difficult) definitions and give really simple interpretations, just as I did for the additive categories, so that you can understand it.

I explained the concept of “rigidity” both geometrically and algebraically. I think the geometry went over better. Rigid means there are no deformations. For example a triangle is rigid but a quadrilateral is not rigid. If you take four sticks and tie them together at the ends then the object you get is flexible or “deformable.” A rectangle is a very special case and is called a “specialization” of the parallelogram and the parallelogram is a “deformation” of the rectangle.

For groups, there are two groups of order 4, the cyclic group and the Klein 4-group:

$$\mathbb{Z}/4, \quad \mathbb{Z}/2 \times \mathbb{Z}/2$$

$\mathbb{Z}/4$ is a “deformation” of $\mathbb{Z}/2 \times \mathbb{Z}/2$. So, $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not “rigid.” Similarly, there are two groups of order 6, the symmetric group and the cyclic group:

$$S_3, \quad \mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$$

However, the symmetric group (permutation group on 3 letters) is non-abelian. And the two abelian (commutative) groups of order 6 are isomorphic (the same). So, commutative groups of order 6 are rigid.

If p is a prime number there is only one group of order p , namely the cyclic group \mathbb{Z}/p . So \mathbb{Z}/p is rigid if p is prime.

The set of (commutative) deformations of $A \times B$ forms a group $\text{Ext}(A, B)$. For example,

$$\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \{0, e\}$$

where 0 represents $\mathbb{Z}/2 \times \mathbb{Z}/2$ and e represents $\mathbb{Z}/4$.

$$\text{Ext}(\mathbb{Z}/3, \mathbb{Z}/2) = 0$$

because there are no commutative deformations of $\mathbb{Z}/3 \times \mathbb{Z}/2$. There is a difference between $\text{Ext}(A, B)$ and $\text{Ext}(B, A)$. They classify different kinds of deformations. An element of $\text{Ext}(A, B)$ is a group E which has a nonzero homomorphism $B \rightarrow E$ and another one $E \rightarrow A$. In the cluster category this means that A must be to the right of B in order for $\text{Ext}(B, A)$ to be nonzero.

7.5.1. definition of clusters.

Definition 7.6. In a cluster category of type A_n , a *cluster* is a set of n distinct indecomposable objects C_1, C_2, \dots, C_n where

$$\text{Ext}(C_i, C_j) = 0$$

for all i, j .

To figure out what the clusters are, we need a formula for $\text{Ext}(A, B)$. This is called either “Serre duality” or “Auslander-Reiten duality” because it was discovered by the late Brandeis University professor Maurice Auslander and his collaborator Idun Reiten.

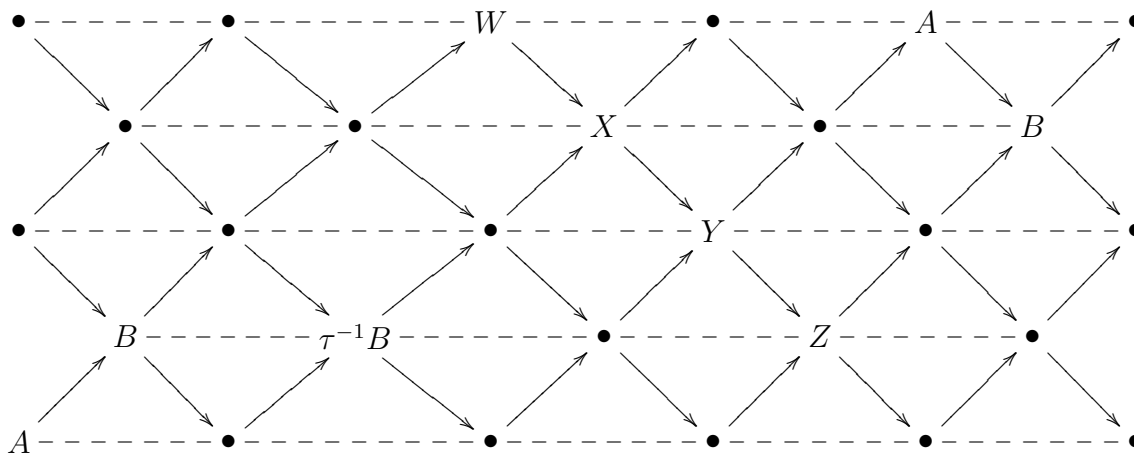
Theorem 7.7. $\text{Ext}(A, B) \cong D \text{Hom}(\tau^{-1}B, A) \cong \text{Hom}(\tau^{-1}B, A)$ where D is vector space dual.

Here τ_{-1} is the inverse of the *Auslander-Reiten translation functor* τ which shifts everything to the left. (Thus τ^{-1} shifts everything to the right.) Note that in the Auslander-Reiten quiver, there is always a dotted line going from τX to X .

The symbol \cong means “isomorphic” although many mathematicians write equality:

$$\text{Ext}(A, B) = D \text{Hom}(\tau^{-1}B, A)$$

7.5.2. *example: A_5 .* First we did calculations on A_5 :



- (1) $\text{Ext}(X, B) = ?$
- (2) $\text{Ext}(W, B) = ?$
- (3) $\text{Ext}(Y, B) = ?$
- (4) $\text{Ext}(Z, B) = ?$

First you use Serre duality:

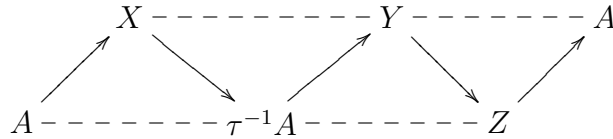
$$\text{Ext}(A, B) \cong \text{Hom}(\tau^{-1}B, A)$$

Then you look at the rectangle starting at $\tau^{-1}B$. Only X, Y are in the rectangle. So,

- (1) $\text{Ext}(X, B) = \text{Hom}(\tau^{-1}B, X) \neq 0$
- (2) $\text{Ext}(W, B) = \text{Hom}(\tau^{-1}B, W) = 0$

- (3) $\text{Ext}(Y, B) = \text{Hom}(\tau^{-1}B, Y) \neq 0$
- (4) $\text{Ext}(Z, B) = \text{Hom}(\tau^{-1}B, Z) = 0$

7.5.3. *example: A_2 .* Next we did A_2 and found all of the clusters. Here is the cluster category:



Since this is A_2 , a cluster consists of two objects C_1, C_2 so that

$$\text{Ext}(C_1, C_2) = 0 = \text{Ext}(C_2, C_1)$$

We took $C_1 = A$ and asked what are the possible C_2 .

- (1) $\text{Ext}(X, A) = \text{Hom}(\tau^{-1}A, X) = 0$
- (2) $\text{Ext}(Y, A) = \text{Hom}(\tau^{-1}A, Y) \neq 0$
- (3) $\text{Ext}(Z, A) = \text{Hom}(\tau^{-1}A, Z) = 0$

In other words, A, X form a cluster and A, Z form a cluster. The objects X, Z are the ones that come right after A and right before A . So, two objects form a cluster if and only if they are consecutive. If we call $B = \tau^{-1}A$ then the clusters are:

$$\{A, X\} \quad \{X, B\} \quad \{B, Y, \} \quad \{Y, Z, \} \quad \{Z, A\}$$

There are $C(3) = 5$ clusters.

7.5.4. *Problems.* 1) Do the same for A_3 .

2) Prove that $\text{Ext}(A, B) \cong \text{Ext}(B, A)$ for any two objects in the cluster category.

8. RESEARCH

When a new area of mathematics opens up, not all of the easy theorems have been discovered. So, students have a chance to find something new. We will try to find something new about clusters and Catalan numbers!