

(4)  $\text{Hom}_{\mathcal{C}}(C, Y) = ?$

First, students realized that  $\text{Hom}_{\mathcal{C}}(D, Z) = 0$  since the only path from  $D$  to  $Z$  is the composition  $g \circ f$  which is equal to zero because of the dotted line from  $D$  to  $Z$ .

Next, we looked at  $\text{Hom}_{\mathcal{C}}(C, Z)$ . There are two paths from  $C$  to  $Z$ . The first is zero since  $g \circ f = 0$  by the previous calculation.

$$\underbrace{g \circ f}_{0} \circ h = 0$$

The other path is

$$g \circ \ell \circ k = g \circ f \circ h = 0$$

which is equal to the previous path and is thus zero by the *mesh relation*

$$f \circ h = \ell \circ k$$

Students got the hang of this and they realized that  $\text{Hom}_{\mathcal{C}}(C, X) = 0$  since all paths from  $C$  to  $X$  are equivalent, by repeated use of mesh relations, to the path

$$m \circ g \circ f \circ h = 0$$

Finally, we came to  $\text{Hom}_{\mathcal{C}}(C, Y)$ . This is nonzero. There are three paths from  $C$  to  $Y$  and they are all equal. But none of these goes through a zero relation. So, they are not necessarily zero. The rule about these diagrams is that all of the information is in the diagram. If a composition of paths is zero, it must be as a consequence of the information in the drawing. This not being the case, the three paths are not zero.

Students asked: Where are the clusters?

It seems that I need one more lecture to show where the clusters enter in this category. This will tie everything together and we can go on to discuss questions and problems.

**7.5. Clusters in the cluster category.** In the cluster category of type  $A_n$ , the clusters are defined to be sets of  $n$  “indecomposable” objects which do not “extend” each other. I will explain the (difficult) definitions and give really simple interpretations, just as I did for the additive categories, so that you can understand it.

I explained the concept of “rigidity” both geometrically and algebraically. I think the geometry went over better. Rigid means there are no deformations. For example a triangle is rigid but a quadrilateral is not rigid. If you take four sticks and tie them together at the ends then the object you get is flexible or “deformable.” A rectangle is a very special case and is called a “specialization” of the parallelogram and the parallelogram is a “deformation” of the rectangle.

For groups, there are two groups of order 4, the cyclic group and the Klein 4-group:

$$\mathbb{Z}/4, \quad \mathbb{Z}/2 \times \mathbb{Z}/2$$

$\mathbb{Z}/4$  is a “deformation” of  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . So,  $\mathbb{Z}/2 \times \mathbb{Z}/2$  is not “rigid.” Similarly, there are two groups of order 6, the symmetric group and the cyclic group:

$$S_3, \quad \mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$$

However, the symmetric group (permutation group on 3 letters) is non-abelian. And the two abelian (commutative) groups of order 6 are isomorphic (the same). So, commutative groups of order 6 are rigid.

If  $p$  is a prime number there is only one group of order  $p$ , namely the cyclic group  $\mathbb{Z}/p$ . So  $\mathbb{Z}/p$  is rigid if  $p$  is prime.

The set of (commutative) deformations of  $A \times B$  forms a group  $\text{Ext}(A, B)$ . For example,

$$\text{Ext}(\mathbb{Z}/2, \mathbb{Z}/2) = \{0, e\}$$

where 0 represents  $\mathbb{Z}/2 \times \mathbb{Z}/2$  and  $e$  represents  $\mathbb{Z}/4$ .

$$\text{Ext}(\mathbb{Z}/3, \mathbb{Z}/2) = 0$$

because there are no commutative deformations of  $\mathbb{Z}/3 \times \mathbb{Z}/2$ . There is a difference between  $\text{Ext}(A, B)$  and  $\text{Ext}(B, A)$ . They classify different kinds of deformations. An element of  $\text{Ext}(A, B)$  is a group  $E$  which has a nonzero homomorphism  $B \rightarrow E$  and another one  $E \rightarrow A$ . In the cluster category this means that  $A$  must be to the right of  $B$  in order for  $\text{Ext}(B, A)$  to be nonzero.

#### 7.5.1. definition of clusters.

**Definition 7.6.** In a cluster category of type  $A_n$ , a *cluster* is a set of  $n$  distinct indecomposable objects  $C_1, C_2, \dots, C_n$  where

$$\text{Ext}(C_i, C_j) = 0$$

for all  $i, j$ .

To figure out what the clusters are, we need a formula for  $\text{Ext}(A, B)$ . This is called either “Serre duality” or “Auslander-Reiten duality” because it was discovered by the late Brandeis University professor Maurice Auslander and his collaborator Idun Reiten.

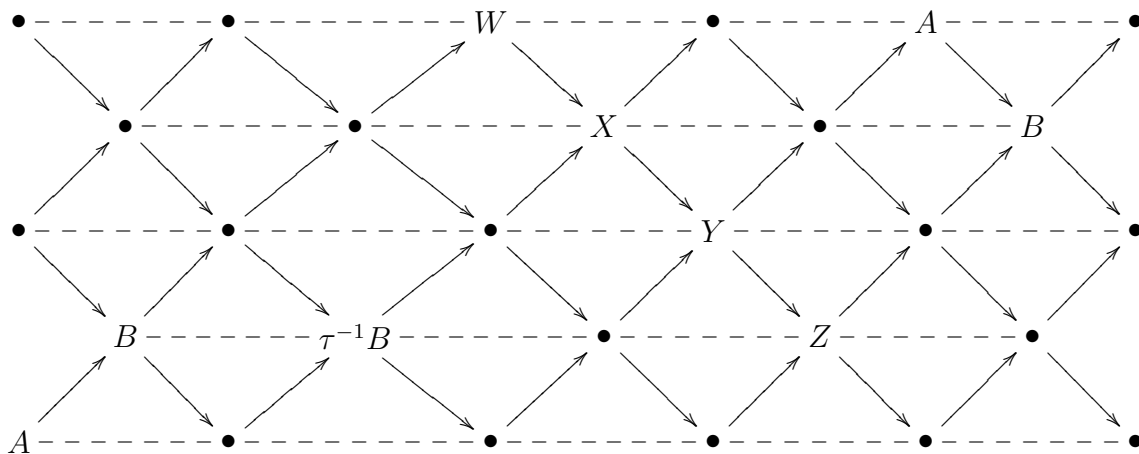
**Theorem 7.7.**  $\text{Ext}(A, B) \cong D \text{Hom}(\tau^{-1}B, A) \cong \text{Hom}(\tau^{-1}B, A)$  where  $D$  is vector space dual.

Here  $\tau^{-1}$  is the inverse of the *Auslander-Reiten translation functor*  $\tau$  which shifts everything to the left. (Thus  $\tau^{-1}$  shifts everything to the right.) Note that in the Auslander-Reiten quiver, there is always a dotted line going from  $\tau X$  to  $X$ .

The symbol  $\cong$  means “isomorphic” although many mathematicians write equality:

$$\text{Ext}(A, B) = D \text{Hom}(\tau^{-1}B, A)$$

7.5.2. *example:  $A_5$ .* First we did calculations on  $A_5$ :



- (1)  $\text{Ext}(X, B) = ?$
- (2)  $\text{Ext}(W, B) = ?$
- (3)  $\text{Ext}(Y, B) = ?$
- (4)  $\text{Ext}(Z, B) = ?$

First you use Serre duality:

$$\text{Ext}(A, B) \cong \text{Hom}(\tau^{-1}B, A)$$

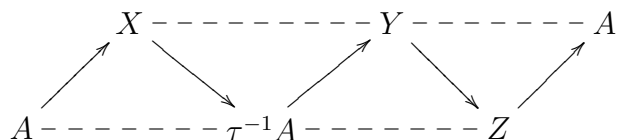
Then you look at the rectangle starting at  $\tau^{-1}B$ . Only  $X, Y$  are in the rectangle. So,

- (1)  $\text{Ext}(X, B) = \text{Hom}(\tau^{-1}B, X) \neq 0$
- (2)  $\text{Ext}(W, B) = \text{Hom}(\tau^{-1}B, W) = 0$

$$(3) \text{Ext}(Y, B) = \text{Hom}(\tau^{-1}B, Y) \neq 0$$

$$(4) \text{Ext}(Z, B) = \text{Hom}(\tau^{-1}B, Z) = 0$$

7.5.3. *example:  $A_2$ .* Next we did  $A_2$  and found all of the clusters. Here is the cluster category:



Since this is  $A_2$ , a cluster consists of two objects  $C_1, C_2$  so that

$$\text{Ext}(C_1, C_2) = 0 = \text{Ext}(C_2, C_1)$$

We took  $C_1 = A$  and asked what are the possible  $C_2$ .

$$(1) \text{Ext}(X, A) = \text{Hom}(\tau^{-1}A, X) = 0$$

$$(2) \text{Ext}(Y, A) = \text{Hom}(\tau^{-1}A, Y) \neq 0$$

$$(3) \text{Ext}(Z, A) = \text{Hom}(\tau^{-1}A, Z) = 0$$

In other words,  $A, X$  form a cluster and  $A, Z$  form a cluster. The objects  $X, Z$  are the ones that come right after  $A$  and right before  $A$ . So, two objects form a cluster if and only if they are consecutive. If we call  $B = \tau^{-1}A$  then the clusters are:

$$\{A, X\} \quad \{X, B\} \quad \{B, Y, \} \quad \{Y, Z, \} \quad \{Z, A\}$$

There are  $C(3) = 5$  clusters.

7.5.4. *Problems.* 1) Do the same for  $A_3$ .

2) Prove that  $\text{Ext}(A, B) \cong \text{Ext}(B, A)$  for any two objects in the cluster category.

## 8. RESEARCH

When a new area of mathematics opens up, not all of the easy theorems have been discovered. So, students have a chance to find something new. We will try to find something new about clusters and Catalan numbers!