FINITE CATEGORIES WITH TWO OBJECTS

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ABSTRACT. In this paper we will examine the possible configurations for finite categories with two objects. In particular, we will look at the case when there is exactly on morphism from B to A.

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INTRODUCTION

Suppose that $\mathcal{C}$ is a category with two objects A, B. Then we want to examine the three separate structures: the endomorphism sets of A and B and the hom set $Hom(A, B)$. Endomorphisms are morphisms from an object to itself. We will examine two kinds of endomorphism sets: those which are groups and those which have annihilators. Then we will look at the consequences for their actions on $Hom(A, B)$. Actually, I won't do that. This is just an example of what you might write which sounds nice. Intod715 should have some overblown hype with promises of interesting reading. Introductions should also have an outline of the contents.

In the first section we will review some basic definitions. In the second section we examine the structure of the endomorphism sets $End(A) = Hom(A, A)$. In the third and last section we see how this affects the structure of the set $Hom(A, B)$.

1. DEFINITIONS

Recall that a *category* consists of "objects" and "morphisms" (also called "arrows" or "homomorphisms") with a composition law which is associative with units. More precisely we have the following. (For more details, see [ML98].)

Definition 1. A *category* $\mathcal{C}$ has a collection $Ob(\mathcal{C})$ of objects, A, B, C, etc and for any two objects A, B a set $Hom(A, B)$ of arrows $f : A \to B$ so that

(1) Composition of arrows $f : A \to B, g : B \to C$ is defined giving an arrow $g \circ f : A \to C$
(2) Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$
(3) Each object A has an identity morphism $id_A : A \to A$ so that $id \circ f = f \circ id = f$.

At this point you could put in an example. You need to put in a short definition and example so that people know what you are talking about. Even if they already know.

Here is a very simple example:

Example 2. Take 3 objects: A, B, C and put in exactly 6 arrows: $A \to A, B \to B, C \to C, A \to B, B \to C, A \to C$. Composition is uniquely determined. So, associativity is automatic. The identity morphisms are the unique arrows from each object to itself.

The first question is: What can you say if there are two morphisms $A \to A$? You can ask rhetorical questions. Then you might need new definitions to make it easier to talk about. Also this gives you a feeling that you are "creating math" or "designing" something. To explore this question we use the definition: If A is an object in a category $\mathcal{C}$, we define an *endomorphism* of A to be an arrow $A \to A$. Note that the set of endomorphisms of A is closed under composition and there is an identity: id_A. We call the set of endomorphisms of A, $End(A)$.

2. Endomorphisms

Note that I haven't actually answered the question. I only made a bunch of definitions and new words.

Suppose that $End(A)$ has two elements. Then one of them is the identity id_A and the other is some other arrow $f : A \to A$. Three of the possible compositions are determined:

$$f \circ id_A = f, \quad id_A \circ f = f, \quad id_A \circ id_A = id_A$$

This leaves only one composition which is not determined: $f \circ f$. There are two choices:

$$f \circ f = f, \quad f \circ f = id_A.$$

Theorem 3. *Both of these are possible.*

Proof. To show that something is possible we just need one example. We need something which we know is a category. The first example is the category of sets and bijections. Since the composition of bijections is a bijection and the identity mapping of a set to itself is a bijection, this is a category. Let A be the set with two elements $A = \{a, b\}$. Then A has two endomorphisms in this category: id_A and f which switches the two elements:

$$f(a) = b, \quad f(b) = a.$$

In this case $f \circ f = id_A$.

Another example is a category in which the functions are labeled with numbers and composition is given by multiplication. Then each object needs to have an endomorphism labeled "1" and the set of numbers needs to be closed under multiplication. In this example, we can define $End(A) = \{0, 1\}$ Since this set of numbers is closed under multiplication and includes the number 1, we get a category with one object and two morphisms. Composition of the nonidentity arrow with itself is

$$0 \cdot 0 = 0$$

So, $f \circ f = f$ is possible. So, both choices for $f \circ f$ are possible. $\square$

Next, suppose that are three endomorphisms of A. Since one of them must be the identity, we have:

$$End(A) = \{id_A, f, g\}.$$

There are four undetermined compositions:

$$f \circ f, \quad g \circ g, \quad f \circ g, \quad g \circ f$$

We could write down all the possibilities. But this would be very tedious and not very instructive. Instead, I will get a more interesting conceptual partial answer. We divide into cases depending on whether multiplication (composition) satisfies cancellation or not.

We say that multiplication by f satisfies *left cancellation* if

$$f \circ x = f \circ y \Rightarrow x = y$$

(in other words, you can cancel f when it is on the left). In this case,

$$f \circ id_A, f \circ f, f \circ g$$

must all be different. In particular, one of these compositions must be equal to id_A. So,

Lemma 4. *If $End(A)$ is finite and $f \in End(A)$ satisfies left cancellation then f has a right inverse (a morphism $h : A \to A$ so that $f \circ h = id_A$.*

If both f and g satisfies left cancellation then all elements will have right inverses making $End(A)$ into a group. But these is only one group of order 3, namely the cyclic group. So, we have the following.

Theorem 5. *If f and g both satisfy left cancellation then they are inverse to each other: $f \circ g = g \circ f = id_A$ and $f \circ f = g, g \circ g = f$.*

In the other extreme, we could have an "annihilator" also called 0.

Definition 6. An *annihilator* is defined to be any endomorphism $0 : A \to A$ so that $0 \circ h = 0 = h \circ 0$ for all endomorphisms h of A.

If A has an annihilator, say $g = 0$, then the remaining endomorphism f has one undetermined composition: $f \circ f$. This could be either $0, id_A$ or f.

Going back to Theorem 3, we note that the two possible structures of the case when $End(A)$ has two elements are

(1) $End(A)$ is a group with two elements. Thus it is the cyclic group $(\mathbb{Z}/2, +)$
(2) $End(A)$ has an identity and an annihilator. In this case $End(A)$ is the multiplicative structure: $(\mathbb{Z}/2, \times)$.

We note that in the case of an additive category, the endomorphism sets have multiplicative structures!

3. Composition

Now we should analyze what happens if there are two objects. (Not today.) Here are some questions and statements.

(1) If A, B have only one endomorphism each and there is a unique morphism $A \to B$ then there is at most one morphism $B \to A$.
(2) If there is only one morphism $B \to A$ and the endomorphism sets $End(A), End(B)$ have two elements each then there could be any number of arrows $A \to B$ is these two endomorphism sets have annihilators.

(3) If there is only one morphism $B \to A$ and the endomorphism sets $End(A), End(B)$ are groups with two elements then there is at most two arrows $A \to B$.
(4) What happens if $End(A)$ is a group and $End(B)$ has an annihilator?
(5) Show that if $\mathcal{C}$ is an additive category with two objects A, B with $End(A), End(B)$ both having exactly two elements, then $Hom(A, B)$ and $Hom(B, A)$ have a power of 2 number of elements each.

References

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.