A stochastic process is a random process which evolves with time. The basic model is the Markov chain. This is a set of “states” together with transition probabilities from one state to another. For example, in simple epidemic models there are only two states: \( S \) = “susceptible” and \( I \) = “infected.” The probability of going from \( S \) to \( I \) increases with the size of \( I \). In the simplest model the \( S \to I \) probability is proportional to \( I \), the \( I \to S \) probability is constant and time is discrete (for example, events happen only once per day). In the corresponding deterministic model we would have a first order recursion.

In a continuous time Markov chain, transition events can occur at any time with a certain probability density. The corresponding deterministic model is a first order differential equation. This includes the “general stochastic epidemic.”

The number of states in a Markov chain is either finite or countably infinite. When the collection of states becomes a continuum, e.g., the price of a stock option, we no longer have a “Markov chain.” We have a more general stochastic process. Under very general conditions we obtain a Wiener process, also known as Brownian motion. The mathematics of hedging implies that stock options should be priced as if they are exactly given by this process. Ito’s formula explains how to calculate (or try to calculate) stochastic integrals which give the long term expected values for a Wiener process.

This course will be a theoretical mathematics course. The goal is to give rigorous mathematical definitions and derive the consequences of simple stochastic models, most of which will be Markov chains. I will not explain the principles of biology, economics and physics behind the models, although I would invite more qualified guest lecturers to explain the background for the models. There will be many examples, not just the ones outlined above.

The prerequisite for the course will be Math 36a (probability using calculus), linear algebra (Math 15) and multivariable calculus (Math 20). Basic linear differential equations will be reviewed at the beginning of the course. Probability will not be reviewed. This is an advanced course for students already familiar with probability. Linear algebra is also heavily used. Statistics is not required.
Outline of course

2. Definitions and general notions about stochastic processes
3. Finite Markov chains
4. Renewal processes
5. Continuous time Markov chains
6. Martingales
7. Wiener processes (Brownian motion)
8. Stochastic integration and Ito’s formula
9. Applications to population biology and epidemiology
10. Application to financial security analysis

Applications will vary according to the interests of students and teacher.

**Required text** *Introduction to Stochastic Processes*, Gregory Lawler, Chapman & Hall

Recommended books:

- *Markov Chains*, J.R. Norris, Cambridge University Press. (This is an excellent book which develops Markov chains in a more leisurely way but does not have stochastic integrals.)

**Grading** 50% homework, 50% in-class performance. Expected grade: A-/B+

There will be weekly homework. The first HW might have the problem: Find a formula for the n-th Fibonacci number by solving the linear recurrence. Students are encouraged to work on their homework in groups and to access all forms of aid including expert advice, internet and other resources. The work you hand in should, however, be in your own words and in your own handwriting. And you should understand what you have written.

In-class activities: “quizzes” will be given every week or other week. Students should form groups of 3 or 4 to work on these problems in class, solve them and help other students in the group to
understand them. Each group should hand in their answers signed by all members of the group. Every student is required to give at least one short oral presentation in front of the class. Attendance is required and counts as part of the grade.

**Students with disability** If you are a student with a documented disability at Brandeis University and if you wish to request a reasonable accommodation for this class, please see the instructor immediately.

**Academic integrity** All members of the Brandeis academic community are expected to maintain the highest standard of academic integrity as outlined in “Rights and Responsibilities.” Any violations will be treated seriously.