MATH 56A: STOCHASTIC PROCESSES
CHAPTER 0

0. Chapter 0

I "reviewed" basic properties of linear differential equations in one variable. I still need to do the theory for several variables.

0.1. **linear differential equations in one variable.**

In the first lecture I discussed linear differential equations in one variable. The problem in degree $n = 2$ is to find a function $y = f(x)$ so that:

$$y'' + ay' + by + c = 0 \tag{0.1}$$

where a, b, c are constants.

The general n-th order linear diff eq in one variable is:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y + a_{n+1} = 0$$

where $a_1, \cdots, a_{n+1}$ are constant.

0.1.1. *associated homogeneous equation.*

The standard procedure is to first take the *associated homogeneous equation* which is given by setting $a_{n+1} = 0$. For the 2nd order equation we get:

$$y'' + ay' + by = 0 \tag{0.2}$$

Lemma 0.1. *If $y = f(x)$ is a solution of the homogeneous equation then so is $y = \alpha f(x)$ for any scalar α.*

Lemma 0.2. *If $y = f_0(x)$ is a solution of the original equation (0.1) and $y = f_1(x)$ is a solution of the homogeneous equation (0.2) then $y = f_0(x) + f_1(x)$ is a solution of the original equation (0.1). Similarly, if f_0, f_1 are solutions of the homogeneous equation then so is $f_0 + f_1$.*

Theorem 0.3. *The set of solutions of the homogenous equation is a vector space of dimension equal to n (the order of the equation).*

This means that if we can find n linearly independent solutions $f_1, f_2, \cdots, f_n$ of the homogeneous equation then the general solution (of the homogenous equation) is a linear combination:

$$y = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$$

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0.1.2. *complete solution of linear diffeqs.* A general solution of the linear diffeq is given by adding a particular solution f_0 to the general solution of the homogeneous equation:

$$y = f_0 + \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$$

The particular solution is easy to find:

$$f_0(x) = -\frac{a_{n+1}}{a_n}$$

if $a_n \neq 0$

$$f_0(x) = - - \frac{a_{n+1}}{a_{n-1}} x$$

if $a_n = 0$ but $a_{n-1} \neq 0$.

The solutions of the homogeneous equations are given by guessing. We just need to find n linearly independent solutions. We guess that $y = e^{\lambda x}$. Then

$$y^k = \lambda^k e^{\lambda x}$$

So, the homogenous equation is:

$$\lambda^n e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + \cdots + a_n e^{\lambda x} = 0$$

Since $e^{\lambda x}$ is never zero we can divide to get:

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

This is a monic (coefficient of λ^n is 1) polynomial of degree n. So, it has n complex roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. If the roots are distinct then the solutions

$$f_1(x) = e^{\lambda_1 x}, f_2(x) = e^{\lambda_2 x}, \cdots$$

are linearly independent and span the solution space.

If roots repeat, e.g., if $\lambda_1 = \lambda_2 = \lambda_3$ then the functions f_2, f_3 are given by

$$f_2(x) = x e^{\lambda_1 x}, \; f_3(x) = x^2 e^{\lambda_1 x},$$

0.1.3. *separation of variables.* Finally, I talked about *separation of variables*. This just means put all the x's on one side of the equation and all the y's on the other side. For example:

$$\frac{dy}{dx} = xy$$

This is a nonlinear diffeq. Separating variables we get:

$$\frac{dy}{y} = x dx$$

Now integrate both sides:

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln y = \frac{x^2}{2} + C$$

Note that there is only one constant. (You get a constant on both sides and C is the difference between the two constants.) The final solution is:

$$y = y_0 exp\left(\frac{x^2}{2}\right)$$

where $y_0 = e^C$.

0.2. **Kermack-McKendrick.** This is from the book *Epidemic Modelling, An Introduction*, D.J. Daley & J.Gani, Cambridge University Press. Kermack-McKendrick is the most common model for the general epidemic. I made two simplifying assumptions:

- the population is homogeneous and
- no births or deaths by other means

Since there are no births, the size of the population N is fixed.

In this model there are three states:

S: = susceptible
I: = infected
R: = removed (immune)

Let $x = \#S, y = \#I, z = \#R$. So

$$N = x + y + z$$

I assume that $z_0 = 0$ (If there are any "removed" people at $t = 0$ we ignore them.)

As time passes, susceptible people become infected and infected recover and become immune. So the size of S decreases and the size of R increases. People move as shown by the arrows:

$$S \longrightarrow I \longrightarrow R$$

The infection rate is given by the *Law of mass action* which says:

The rate of interaction between two different subsets of the population is proportional to the product of the number of elements in each subset.

So,

$$\frac{dx}{dt} = -\beta xy$$

where β is a positive constant.

Infected people recover at a constant rate:

$$\frac{dz}{dt} = \gamma y$$

where γ is a positive constant.

This is a system of nonlinear equations. To solve it we make it linear by dividing:

$$\frac{dx}{dz} = \frac{dx/dt}{dz/dt} = \frac{-\beta xy}{\gamma y} = \frac{-\beta x}{\gamma}$$

This is a linear differential equation with solution

$$x = x_0 exp\left(\frac{-\beta}{\gamma} z\right) = x_0 e^{-z/\rho}$$

where $\rho = \gamma/\beta$ is the *threshold* population size.

Since $x + y + z$ is fixed we can find y:

$$y = N - x - z = N - x_0 e^{-z/\rho} - z$$

Differentiating gives:

$$\frac{dy}{dz} = \frac{x_0}{\rho} e^{-z/\rho} - 1$$

$$\frac{d^2y}{dz^2} = -\frac{x_0}{\rho^2} e^{-z/\rho} < 0$$

So, the function is concave down with initial slope

$$\frac{dy}{dz} = \frac{x_0}{\rho} - 1$$

which is positive if and only if $x_0 > \rho$. So, according to this model, the number of infected will initially increase if and only if the number of susceptibles is greater than the threshold value $\rho = \gamma/\beta$.

By plotting the graphs of the functions we also see that the infection will taper off with a certain number of susceptibles x_∞ who never get infected.

0.3. **systems of first order equations.** I explained that differential equations involving second and higher order derivatives can be reduced to a system of first order equations by introducing more variables. Then I did the following example.

$$y' = z$$

$$z' = 6y - z$$

In matrix form this is:

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

Which we can write as $Y = AY$ with

$$Y = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$$

0.3.1. *exponential of a matrix.* The solution of this equations is

$$Y = e^{tA}Y_0$$

where $Y_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ and

$$e^{tA} := I_2 + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots$$

This works because the derivative of each term is A times the previous term:

$$\frac{d}{dt}\frac{t^kA^k}{k!} = \frac{kt^{k-1}A^k}{k!} = \frac{t^{k-1}A^k}{(k-1)!} = A\frac{t^{k-1}A^{k-1}}{(k-1)!}$$

So,

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

0.3.2. *diagonalization (corrected).* Then I explained how to compute r^{tA}. You have to *diagonalize* A. This means

$$A = QDQ^{-1}$$

where D is a diagonal matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

I should have explained this formula so that I get it right: If X_1, X_2 are eigenvalues of A with eigenvalues d_1, d_2 then $AX_1 = X_1d_1, AX_2 = X_2d_2$ and

$$A(X_1X_2) = (X_1d_1X_2d_2) = (X_1X_2)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Solve for A gives

$$A = (X_1 X_2) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} (X_1 X_2)^{-1} = QDQ^{-1}$$

where $Q = (X_1, X_2)$.

This is a good idea because $A^2 = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1}$ and more generally, $t^k A^k = t^k Q D^k Q^{-1}$. Divide by $k!$ and sum over k to get:

$$e^{tA} = Qe^{tD}Q^{-1} = Q \begin{pmatrix} e^{td_1} & 0 \\ 0 & e^{td_2} \end{pmatrix} Q^{-1}$$

0.3.3. *eigenvectors and eigenvalues.* The diagonal entries d_1, d_2 are the eigenvalues of the matrix A and $Q = (X_1 X_2)$ where X_i is the eigenvector corresponding to d_i. This works if the eigenvalues of A are distinct. The eigenvalues are defined to be the solutions of the equation

$$\det(A - \lambda I) = 0$$

but there is a trick to use for 2×2 matrices. The determinant of a matrix is always the product of its eigenvalues:

$$\det A = d_1 d_2 = -6$$

The trace (sum of diagonal entries) is equal to the sum of the eigenvalues:

$$trA = d_1 + d_2 = -1$$

So, $d_1 = 2, d_2 = -3$. The eigenvalues are $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. So

$$Q = (X_1 X_2) = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}$$

The solution to the original equation is

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

0.4. **Linear difference equations.** We are looking for a sequence of numbers $f(n)$ where n ranges over all the integers from K to N ($K \leq n \leq N$) so that

$$f(n) = af(n-1) + bf(n+1) \tag{0.3}$$

I pointed out that the solution set is a vector space of dimension 2. So we just have to find two linearly independent solutions. Then I followed the book.

The solution has the form $f(n) = c^n$ where you have to solve for c:

$$c^n = ac^{n-1} + bc^{n+1}$$

$$bc^2 - c + a = 0 \tag{0.4}$$

$$c = \frac{1 \pm \sqrt{1-4ab}}{2b}$$

There were two cases.

Case 1: ($4ab \neq 1$) When the quadratic equation (0.4) has two roots C_1, c_2 then the linear combinations of c_1^n and c_2^n give all the solutions of the homogeneous linear recursion (0.3).

Case 2: ($4ab = 1$) In this case there is only one root $c = \frac{1}{2b}$ and the two independent solutions are $f(n) = c^n$ and nc^n. The reason we get a factor of n is because when a linear equation has a double root then this root will also be a root of the derivative. This gives $f(n) = nc^{n-1}$ as a solution. But then you can multiply by the constant c since the equation is homogeneous.

Example 0.4. *(Fibonacci numbers) These are given by* $f(0) = 1, f(1) = 1$ *and* $f(n+1) = f(n) + f(n-1)$ *or:*

$$f(n) = f(n+1) - f(n-1)$$

This is $a = -1, b = 1$*. The roots of the quadratic equation are* $c = \frac{1\pm\sqrt{5}}{2}$*. So,*

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is a rational number since it is Galois invariant (does not change if you switch the sign of $\sqrt{5}$*). However, it is not clear from the formula why it is an integer.*