0.3. **systems of first order equations.** I explained that differential equations involving second and higher order derivatives can be reduced to a system of first order equations by introducing more variables. Then I did the following example.

$$y' = z$$

$$z' = 6y - z$$

In matrix form this is:

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

Which we can write as $Y = AY$ with

$$Y = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$$

0.3.1. *exponential of a matrix.* The solution of this equations is

$$Y = e^{tA} Y_0$$

where $Y_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ and

$$e^{tA} := I_2 + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots$$

This works because the derivative of each term is A times the previous term:

$$\frac{d}{dt}\frac{t^kA^k}{k!} = \frac{kt^{k-1}A^k}{k!} = \frac{t^{k-1}A^k}{(k-1)!} = A\frac{t^{k-1}A^{k-1}}{(k-1)!}$$

So,

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

0.3.2. *diagonalization (corrected).* Then I explained how to compute r^{tA}. You have to *diagonalize* A. This means

$$A = QDQ^{-1}$$

where D is a diagonal matrix $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

I should have explained this formula so that I get it right: If X_1, X_2 are eigenvalues of A with eigenvalues d_1, d_2 then $AX_1 = X_1d_1, AX_2 = X_2d_2$ and

$$A(X_1X_2) = (X_1d_1X_2d_2) = (X_1X_2)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

Solve for A gives

$$A = (X_1X_2)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}(X_1X_2)^{-1} = QDQ^{-1}$$

where $Q = (X_1, X_2)$.

This is a good idea because $A^2 = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1}$ and more generally, $t^kA^k = t^kQD^kQ^{-1}$. Divide by $k!$ and sum over k to get:

$$e^{tA} = Qe^{tD}Q^{-1} = Q\begin{pmatrix} e^{td_1} & 0 \\ 0 & e^{td_2} \end{pmatrix}Q^{-1}$$

0.3.3. *eigenvectors and eigenvalues.* The diagonal entries d_1, d_2 are the eigenvalues of the matrix A and $Q = (X_1X_2)$ where X_i is the eigenvector corresponding to d_i. This works if the eigenvalues of A are distinct. The eigenvalues are defined to be the solutions of the equation

$$\det(A - \lambda I) = 0$$

but there is a trick to use for 2×2 matrices. The determinant of a matrix is always the product of its eigenvalues:

$$\det A = d_1d_2 = -6$$

The trace (sum of diagonal entries) is equal to the sum of the eigenvalues:

$$trA = d_1 + d_2 = -1$$

So, $d_1 = 2, d_2 = -3$. The eigenvalues are $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. So

$$Q = (X_1X_2) = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\det Q}\begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}$$

The solution to the original equation is

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

0.4. **Linear difference equations.** We are looking for a sequence of numbers $f(n)$ where n ranges over all the integers from K to N ($K \leq n \leq N$) so that

$$f(n) = af(n-1) + bf(n+1) \tag{0.3}$$

I pointed out that the solution set is a vector space of dimension 2. So we just have to find two linearly independent solutions. Then I followed the book.

The solution has the form $f(n) = c^n$ where you have to solve for c:

$$c^n = ac^{n-1} + bc^{n+1}$$

$$bc^2 - c + a = 0 \tag{0.4}$$

$$c = \frac{1 \pm \sqrt{1-4ab}}{2b}$$

There were two cases.

Case 1: ($4ab \neq 1$) When the quadratic equation (0.4) has two roots C_1, c_2 then the linear combinations of c_1^n and c_2^n give all the solutions of the homogeneous linear recursion (0.3).

Case 2: ($4ab = 1$) In this case there is only one root $c = \frac{1}{2b}$ and the two independent solutions are $f(n) = c^n$ and nc^n. The reason we get a factor of n is because when a linear equation has a double root then this root will also be a root of the derivative. This gives $f(n) = nc^{n-1}$ as a solution. But then you can multiply by the constant c since the equation is homogeneous.

Example 0.4. *(Fibonacci numbers) These are given by* $f(0) = 1, f(1) = 1$ *and* $f(n+1) = f(n) + f(n-1)$ *or:*

$$f(n) = f(n+1) - f(n-1)$$

This is $a = -1, b = 1$. *The roots of the quadratic equation are* $c = \frac{1\pm\sqrt{5}}{2}$. *So,*

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is a rational number since it is Galois invariant (does not change if you switch the sign of $\sqrt{5}$*). However, it is not clear from the formula why it is an integer.*