MATH 56A: STOCHASTIC PROCESSES
CHAPTER 2

2. Countable Markov Chains

I started Chapter 2 which talks about Markov chains with a countably infinite number of states. I did my favorite example which is on page 53 of the book.

2.1. **Extinction probability.** In this example we consider a population of one cell creatures which reproduce asexually. At each time interval, each creature produces X offspring and then dies. Here X is a random variable equal to $0, 1, 2, 3$ etc with probabilities $p_k = \mathbb{P}(X = k)$.

You can read the complete analysis in the book, but what I explained in class was the most striking part: if the average number of offspring is equal to 1 (and it is somewhat random, i.e., not always equal to 1) then the probability of extinction of the population is one.

In this model the state space is the set of nonnegative integers. If the state is i then there are i creature in the population. At each time interval each of these creatures dies and produces a random number of offspring. The state 0 is an absorbing state. We want to know the probability that we eventually land in that state. We start with the definition:

$$a := \mathbb{P}(X_n = 0 \text{ eventually} | X_0 = 1)$$

So, a is the probability that the species eventually becomes extinct if it starts out with exactly one creature. The theorem is that $a = 1$. This implies that population dies out if we start with any number of creatures. The reason is that, because of asexual reproduction, the descendants of each individual will not mix and so, *assuming independence of the probabilities of extinction* for each "family," we get that

$$a^k = \mathbb{P}(X_n = 0 \text{ eventually} | X_0 = k)$$

The point is that, if $a = 1$ then $a^k = 1$ for any $k \geq 0$.

To calculate a we look at what happens after one time interval.

$$a = \sum_k \mathbb{P}(X_n = 0 \text{ eventually} | X_1 = k)\mathbb{P}(X_1 = k) = \sum_k a^k p_k$$

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But this is the *generating function* for p_k which is defined by

$$\phi(s) := \mathbb{E}(s^X) = \sum_k s^k \mathbb{P}(X = k) = \sum_k s^k p_k$$

The extinction probability is equal to the generating function!!

$$a = \phi(a)$$

The generating function has the property that $\phi(1) = 1$. Here is the proof:

$$\phi(1) = \sum_k 1^k p_k = \sum_k p_k = 1$$

The derivative of $\phi(s)$ is

$$\phi'(s) = \sum_k k s^{k-1} p_k \tag{2.1}$$

If we put $s = 1$ we get the expected value of X (the number of offspring) which we are assuming is equal to 1.

$$\phi'(1) = \sum_k k p_k = \mathbb{E}(X) = 1$$

The second derivative is

$$\phi''(s) = \sum_{k \geq 2} k(k-1) s^{k-2} p_k > 0$$

This is greater than zero (for all s) if $p_2 > 0$ or $p_k > 0$ for some $k \geq 2$. But, if X has average 1 and is not always equal to 1 then it must be sometimes more and sometimes less than 1. So, there is a positive probability that $X \geq 2$. So, $\phi''(s) > 0$.

Now, graph the function $y = \phi(s)$. Since $\phi(1) = 1$ and $\phi'(1) = 1$, the graph goes through the point $(1, 1)$ with slope 1. Since it is concave up, it has to curve away from the line $y = s$ on both sides of that point. So, the only solution to the equation $a = \phi(a)$ is $a = 1$.

For the general analysis we need the following lemma.

Lemma 2.1. *a is the smallest nonnegative solution to the equation $a = \phi(a)$.*

Proof. The eventual extinction probability is a limit of finite extinction probabilities:

$$a = \lim_{n \to \infty} a_n$$

where

$$a_n = \mathbb{P}(X_n = 0 | X_0 = 1)$$

These finite extinction probabilities are calculated recursively as follows:

$$a_n = \sum_k \mathbb{P}(X_n = 0|X_1 = k)\mathbb{P}(X_1 = k|X_0 = 1)$$

$$= \sum_k \mathbb{P}(X_{n-1} = 0|X_0 = k)\mathbb{P}(X_1 = k|X_0 = 1) = \sum_k a_{n-1}^k p_k = \phi(a_{n-1})$$

Let $\hat{a}$ be the smallest nonnegative real number so that $\hat{a} = \phi(\hat{a})$. Then we just have to show that $a_n \leq \hat{a}$ for every $n \geq 0$. This is true for $n = 0$ since

$$a_0 = \mathbb{P}(X_0 = 0|X_0 = 1) = 0 \leq \hat{a}$$

Suppose by induction that $a_{n-1} \leq \hat{a}$. Then we have to show that $a_n \leq \hat{a}$. But, if you look at the equation (2.1) you see that $\phi'(s) \geq 0$. So,

$$a_n = \phi(a_{n-1}) \leq \phi(\hat{a}) = \hat{a}$$

Therefore, $a_n \leq \hat{a}$ for all $n \geq 0$. So, $a \leq \hat{a}$. So, $a = \hat{a}$. □

Theorem 2.2. *If $\mathbb{E}(X) > 1$ then the probability of extinction is less than 1. If $\mathbb{E}(X) \leq 1$ then the probability of extinction is one, except in the case when the population is constant (i.e., when $p_1 = 1$).*

Proof. By the lemma, the extinction probability a is the first point of intersection of the graph of $y = \phi(s)$ with the graph of $y = s$. But $\phi(s)$ goes through the point $(1, 1)$ with slope $\mathbb{E}(X)$ and is always concave up. A drawing of the graphs proves the theorem. □

2.2. **Random walk in an integer lattice.** Today I want to explain the proof that the simple random walk in the lattice $\mathbb{Z}^d$ is recurrent if $d \leq 2$ and transient if $d \geq 3$.

2.2.1. *One dimensional case.* First look at the case $d = 1$. Then the state space is $\mathbb{Z}$ the set of all integers. At each step we go either left or right with probability $\frac{1}{2}$. This is periodic with period 2. We need an even number of steps to get back to the starting point. Let's say the starting point is 0.

$$p_{2n}(0,0) = C(2n,n)\left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{n!n!}\left(\frac{1}{2}\right)^{2n}$$

Use Stirling's formula:

$$n! \sim e^{-n}n^n\sqrt{2\pi n}$$

where $\sim$ means the ratio of the two sides converges to 1 as $n \to \infty$. So,

$$(2n)! \sim e^{2n}(2n)^{2n}\sqrt{4\pi n}$$

and

$$\frac{(2n)!}{n!n!} \sim \frac{e^{2n}(2n)^{2n}\sqrt{4\pi n}}{e^{2n}n^{2n}2\pi n} = \frac{2^{2n}}{\sqrt{\pi n}}$$

which means that

$$p_{2n}(0,0) \sim \frac{1}{\sqrt{\pi n}}$$

The expected number of return visits to 0 is

$$\sum_{n=1}^{\infty} p_{2n}(0,0) \approx \sum \frac{1}{\sqrt{\pi n}} = \infty$$

So, 0 is recurrent in $\mathbb{Z}$. Since there is only one communication class, all states are recurrent.

2.2.2. *Higher dimensions.* In the planar lattice $\mathbb{Z}^2$, both coordinates must be 0 at the same time in order for the particle to return to the origin. Therefore,

$$p_{2n}(0,0) \sim \frac{1}{\pi n}$$

and

$$\sum_{n=1}^{\infty} p_{2n}(0,0) \approx \sum \frac{1}{\pi n} = \infty$$

and $\mathbb{Z}^2$ is recurrent.

When $d > 2$ we get

$$p_{2n}(0,0) \sim \frac{1}{(\pi n)^{d/2}}$$

and

$$\sum_{n=1}^{\infty} p_{2n}(0,0) \approx \sum \frac{1}{(\pi n)^{d/2}}$$

which converges by the integral test. So, the expected number of visits is finite and $\mathbb{Z}^3, \mathbb{Z}^4$, etc are transient.

2.2.3. *Stirling's formula.* Exercise 2.18 on page 62 gives a rigorous proof of Stirling's formula. I will give a simpler proof which misquotes (!) the central limit theorem. It starts with Y_n the Poisson random variable with mean n. It has probability distribution

$$\mathbb{P}(Y_n = k) = e^{-n}\frac{n^k}{k!}$$

The variance of a Poisson variable is equal to its mean. So, the standard deviation is $\sqrt{n}$. The central limit theorem says that, as $n \to \infty$ the Poisson distribution is approximated by the normal distribution $N(n, \sqrt{n})$ with $\mu = n, \sigma = \sqrt{n}$. But Poisson is discrete so you have to take integer steps:

$$\mathbb{P}(Y_n = n) \sim \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} N(n, \sqrt{n}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} N(0, \sqrt{n})$$

$$e^{-n}\frac{n^n}{n!} \sim \frac{1}{\sqrt{2\pi n}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-x^2/n}\, dx$$

But,

$$\lim_{n\to\infty} \int_0^1 e^{-x^2/n}\, dx = \int_0^1 e^0\, dx = 1$$

So,

$$e^{-n}\frac{n^n}{n!} \sim \frac{1}{\sqrt{2\pi n}}$$

This is Stirling's formula when you solve for $n!$.

Note: the central limit theorem does not actually give approximations for single values of discrete probability distributions, it only approximates *sums* of values over a range of values which is a fixed multiple of the standard deviation. However, the book points out that the Poisson distribution is fairly uniform in its values. So, the sum over a range is approximated by a single value times the size of the range. (This is the point of part (b) of the exercise.)

2.3. **Transient-recurrent.** If X_n is an irreducible aperiodic Markov chain then there are 3 possibilities:

(1) X_n is transient
(2) X_n is null recurrent
(3) X_n is positive recurrent

What do these mean and how can you tell which category you are in?

First of all, in the finite case, you always have positive recurrence. Null recurrence is something that happens only in the infinite case.

To tell whether X_n is transient or recurrent we look at the function $\alpha(x)$ defined as follows.

Fix a state $z \in S$. (S is the set of all states. $|S| = \infty$.)

Take $\alpha(x)$ to be the probability that you go from x to z. I.e., you start at x and see whether you are ever in state z:

$$\alpha(x) := \mathbb{P}(X_n = z \text{ for some } n \geq 0 \,|\, X_0 = x)$$

This function satisfies three equations:

(1) $0 \leq \alpha(x) \leq 1$ (since $\alpha(x)$ is a probability)
(2) $\alpha(z) = 1$ (since you start at $x = z$)
(3) If $x \neq z$ then

$$\alpha(x) = \sum_{y \in S} p(x,y)\alpha(y)$$

(To get from $x \neq z$ to z you take one step to y and then go from y to z.)

The above three equations are always true. The next equation tells us whether the chain is transient or recurrent.

(4) $\inf_x \alpha(x) = 0$ iff X_n is transient
(5) $\alpha(x) = 1$ for all $x \in S$ iff X_n is recurrent.

Theorem 2.3. *$\alpha(x)$ is the smallest solution of equations (1),(2),(3). I.e., if $\hat{\alpha}(x)$ is another solution then*

$$\alpha(x) \leq \hat{\alpha}(x)$$

for all $x \in S$.

Remark 2.4. *I pointed out in class that $\alpha(x) = 1$ is always a solution of equations (1),(2),(3) since (3), in matrix form is*

$$A = PA$$

I.e., $A = (\alpha(x))$ is a right eigenvector of P with eigenvalue 1.

Proof. $\alpha(x) = \lim_{n\to\infty} \alpha_n(x)$ where $\alpha_n(x)$ is the probability that you get from x to z in n steps or less:

$$\alpha_n(x) := \mathbb{P}(X_k = z \text{ for some } 0 \le k \le n \,|\, X_0 = x)$$

Then, I claim that

$$\alpha_n(x) \le \hat{\alpha}(x) \tag{2.2}$$

for all n and all x. This is true for $n = 0$ since $\alpha_0(x) = 0$ for all $x \neq z$. Suppose that (2.2) holds for $n - 1$. Then

$$\alpha_n(x) = \sum_y p(x, y)\alpha_{n-1}(y) \le \sum_y p(x, y)\hat{\alpha}(y) = \hat{\alpha}(x)$$

So, (2.2) holds for n. By induction it holds for all n. So, $\alpha(x) = \lim \alpha_n(x) \le \hat{\alpha}(x)$. □

Corollary 2.5. *Given that X_n is irreducible and aperiodic, X_n is transient iff there is a solution $\hat{\alpha}(x)$ of equations (1)-(4).*

Proof. The real probability $\alpha(x)$ is $\le \hat{\alpha}(x)$. So, $0 \le \alpha(x) \le \hat{\alpha}(x) \to 0$ implies that $\alpha(x) \to 0$. So, X_n is transient. □

2.4. **example.** Take a random walk on $S = \{0, 1, 2, 3, \cdots\}$ with partially reflecting wall at 0. So, the probability of going left (or standing still at 0) is $p > 0$ and the probability of going right is $q = 1 - p > 0$:

$$p(n, n+1) = q, \quad p(n, n-1) = p, \quad p(0, 0) = p$$

Let $z = 0$. We want to find the smallest solution $\alpha(n)$ of (1),(2),(3). But we already know how to do this. Equation (3) says:

$$\alpha(n) = p\alpha(n-1) + q\alpha(n+1)$$

The solution is $\alpha(x) = r^n$ where

$$r = p + qr^2$$

So,

$$r = \frac{1 \pm \sqrt{1-4pq}}{2q} = \frac{1 - \sqrt{1-4pq}}{2q}$$

(We want the smaller solution.)

2.4.1. $p = 1/2$. In this case $r = 1$. So, $\alpha(n) = r^n = 1$ and the Markov chain is recurrent. In fact we will see that it is null recurrent.

2.4.2. $p < 1/2$. In this case we can write

$$p = \frac{1}{2} - \epsilon, \quad pq = \frac{1}{4} - \epsilon^2$$
$$q = \frac{1}{2} + \epsilon, \quad 4pq = 1 - 4\epsilon^2$$
$$2q = 1 + 2\epsilon, \quad 1 - 4pq = 4\epsilon^2$$
$$r = \frac{1 - 2\epsilon}{1 + 2\epsilon} < 1$$

So, $\alpha(n) = r^n \to 0$ and X_n is transient.

2.4.3. $p > 1/2$. This is the part I skipped in class.

$$p = \frac{1}{2} + \epsilon, \quad pq = \frac{1}{4} - \epsilon^2$$
$$q = \frac{1}{2} - \epsilon, \quad 4pq = 1 - 4\epsilon^2$$
$$2q = 1 - 2\epsilon, \quad 1 - 4pq = 4\epsilon^2$$
$$r = \frac{1 - 2\epsilon}{1 - 2\epsilon} = 1$$

So, $\alpha(n) = r^n = 1$ for all n and X_n is recurrent.

2.5. **Null recurrence-positive recurrence.**

Definition 2.6. *An irreducible aperiodic Markov chain is called* null recurrent *if it is recurrent but*

$$\lim_{n\to\infty} p_n(x, y) = 0$$

for all states x, y. It is called positive recurrent *if it is recurrent but not null recurrent.*

Theorem 2.7. *If a Markov chain is positive recurrent then*

$$\lim_{n\to\infty} p_n(x, y) = \pi(y) > 0$$

is an invariant probability distribution. Also,

$$\mathbb{E}(T) = \frac{1}{\pi(y)}$$

where T is the first return time to y:

$$T = \text{ smallest } n > 0 \text{ so that } X_n = y \text{ given that } X_0 = y$$

Remember that an invariant distribution is a left eigenvector of P:

$$\pi P = \pi$$

with $\sum \pi(x) = 1$.

Corollary 2.8. *There is an invariant probability distribution π if and only if the Markov chain is positive recurrent.*

Proof. Since π is invariant,

$$\sum_x \pi(x) p_n(x,y) = \pi(y)$$

But $\pi(y)$ is positive and constant (does not change as we increase n). Therefore, the probabilities $p_n(x,y)$ cannot all go to zero and X_n cannot be null recurrent. $\square$

2.6. **example, continued.** Going back to the random walk on $S = \{0,1,2,3,\cdots\}$ with partially reflecting wall at 0, the definition of invariant distribution says:

$$\pi(y) = \sum_x \pi(x) p(x,y)$$

In the random walk this is:

$$\pi(n) = q\pi(n-1) + p\pi(n+1) \tag{2.3}$$

which has solution

$$\pi(n) = \frac{r^n}{1-r}$$

(We have to divide by $1 - r = 1 + r + r^2 + \cdots$ so that $\sum \pi(n) = 1$.)

$$r = \frac{1 - \sqrt{1-4pq}}{2p}$$

If $p = \frac{1}{2} + \epsilon$ then $q = \frac{1}{2} - \epsilon$ and

$$r = \frac{1-2\epsilon}{1+2\epsilon}, \quad 1 - r = \frac{4\epsilon}{1+2\epsilon}$$

So, we get an invariant distribution:

$$\pi(n) = \frac{r^n}{1-r} = \frac{1-2\epsilon}{4\epsilon}$$

Therefore, the chain is positive recurrent if $p > 1/2$.

If $p = 1/2$ then $r = 1$ and $\pi(n) = 1$ or $\pi(n) = n$ and neither solution can be normalized (scaled so that the sum is 1). Therefore, X_n is null recurrent if $p = 1/2$.

2.7. **Chart.** An irreducible aperiodic Markov chain has three possible types of behavior:

	$\lim_{n\to\infty} p_n(x,y)$	$\mathbb{E}(T)$	$\mathbb{P}(T<\infty)$	$\alpha(x)$
transient	$=0$	$=\infty$	<1	$\inf \alpha(x)=0$
0 *recurrent*	$=0$	$=\infty$	$=1$	$\alpha(x)=1\ \forall x$
+ *recurrent*	$=\pi(y)>0$	$=\frac{1}{\pi(y)}<\infty$	$=1$	$\alpha(x)=1\ \forall x$

In the transient case, $\mathbb{P}(T<\infty)<1$ is the same as $\mathbb{P}(T=\infty)>0$. This implies transient because it says that there is a chance that you never return. If a guy keeps leaving and there is a chance that he doesn't return each time then eventually, with probability one, he will not return. So, the number of visits if finite. If $\mathbb{P}(T=\infty)=0$ then he always returns and so he will keep coming back an infinite number of times. Since $\mathbb{P}(T=\infty)$ is either 0 or positive, *this proves that we either have transience or recurrence!!*