

**MATH 56A: STOCHASTIC PROCESSES**  
**CHAPTER 6**

6. RENEWAL

Mathematically, renewal refers to a continuous time stochastic process with states  $0, 1, 2, \dots$

$$N_t \in \{0, 1, 2, 3, \dots\}$$

so that you only have jumps from  $x$  to  $x + 1$  and the probability of jumping from  $x$  to  $x + 1$  depends only on how long the process was at state  $x$ . *Renewal* occurs at each jump.

$N_t :=$  number of jumps that occur in time interval  $(0, t]$

The jumps (renewals) occur at times  $Y, Y + T_1, Y + T_1 + T_2$ , etc. and

$$Y + T_1 + \dots + T_n = \inf\{t \mid N_t = n + 1\}$$

The interpretation is that there is an object or process which lasts for a certain amount of time which is random. When the object dies or the process stops then you replace the object with a new one or you restart the process from the beginning: You “renew” the process each time it stops. The number  $N_t$  is equal to the number of times renewal has taken place up to time  $t$ .  $Y$  is the lifetime of the initial process,  $T_1$  is the lifetime of the second one,  $T_2$  is the lifetime of the third one, etc. If the first process starts from the beginning then  $Y = 0$  and the numbering is different.  $T_n$  becomes the lifetime of the  $n$ th process:

$$T_1 + \dots + T_n = \inf\{t \mid N_t = n\}$$

I gave a light bulb as an example. There are three kinds of light bulbs:

- (1) The guaranteed light bulb which will last exactly 1000 hours.
- (2) The Poisson light bulb. This light bulb is as good as new as long as it is working. Assume it has an expected life of 1000 hours. ( $\lambda = 1/1000$ ).
- (3) A general light bulb which has a general probability distribution with the property that its expected life is 1000 hours.

In all three cases,

$$\mu = \mathbb{E}(T) = 1000$$

where  $T$  is the length of time that the light bulb lasts.

The first question is: Which light bulb is worth more? The answer is that they are all worth the same. They all give an expected utility of 1000 hours of light. With the general light bulb, there is another question: How long do you expect the last light bulb to last after it has been used for a certain amount of time? This depends on the light bulb. For example, if the guaranteed light bulb has been used for 500 hours then it is only worth half as much as a new one. If the Poisson light bulb lasts 500 hours then it is still worth the same as a new one. We will look at the value of a general light bulb (or a renewal process with a general distribution.)

**6.1. Renewal theorem.** The guaranteed light bulb is an example of a *periodic* renewal process. Each renewal occurs at multiples of 1000 hours.

**Definition 6.1.** *A renewal process is periodic if renewals always occur at (random) integer multiples of a fixed time interval  $\Delta t$  starting with the first renewal which occurs at time  $Y$ .*

The renewal theorem says that, if renewal is not periodic, then the occurrences of the renewal will be spread out evenly around the clock. The probability that it will occur will depend only on the length of time you wait. Since the average waiting time is  $\mu$ , the probability is approximately the proportion of  $\mu$  that you wait:  $\mathbb{P} \cong \Delta t / \mu$ .

For the lightbulb, suppose you install a million lightbulbs at the same time. Then after a while the number of light bulbs that burn out each day will be constant. This (after dividing by one million) will be the *equilibrium distribution*.

**Theorem 6.2** (Renewal Theorem). *If a renewal process is aperiodic then, as  $t \rightarrow \infty$ ,*

$$\mathbb{P}(\text{renewal occurs in time } (t, t + dt]) \rightarrow \frac{dt}{\mu}$$

where  $\mu = \mathbb{E}(T)$ . This is equivalent to saying that

$$\lim_{t \rightarrow \infty} \mathbb{E}(\text{number of renewals in time } (t, t + s]) = \frac{s}{\mu}$$

$$\lim_{t \rightarrow \infty} \mathbb{E}(N_{t+s} - N_t) = \frac{s}{\mu}$$

**6.2. age of current process.** At any time  $t$ , let  $A_t$  be the life of the current process. This would be the answer to the question: How long ago did you replace the light bulb? The book says that the pair  $(N_t, A_t)$  determines the future of the process.  $B_t$  denotes the remaining life of the current process. (How long will the current light bulb last?) First I needed the following lemma.

6.2.1. *picture for an expected value.*

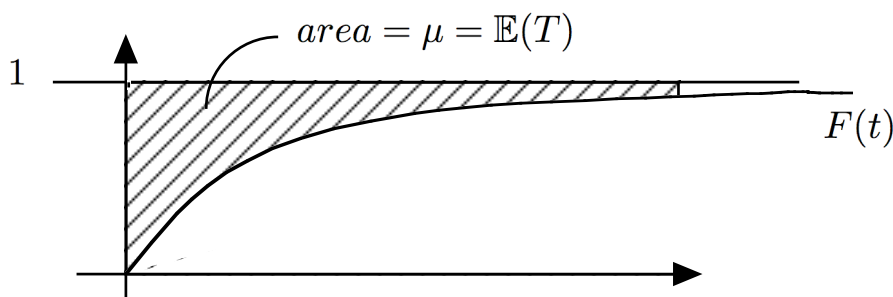


FIGURE 1. The shaded area above the distribution function  $F(t)$  is equal to the expectation.

**Lemma 6.3.** *If  $T \geq 0$  is a nonnegative random variable then the expected value of  $T$  is given by*

$$\mathbb{E}(T) = \int_0^{\infty} 1 - F(t) dt$$

*Proof.* The expected value of  $T$  is defined by the integral

$$\mathbb{E}(T) = \int_0^{\infty} tF(t) dt$$

Substituting the integral

$$t = \int_0^t ds = \int_{0 \leq s \leq t} ds$$

we get:

$$\mathbb{E}(T) = \iint_{0 \leq s \leq t} f(t) ds dt$$

On the other hand,

$$1 - F(s) = \mathbb{P}(T > s) = \int_s^{\infty} f(t) dt$$



For the case of the exponential distribution we have  $1 - F(t) = e^{-\lambda t}$  and  $\mu = 1/\lambda$ . So

$$f(t) = \lambda e^{-\lambda t} = \lambda(1 - F(t)) = \frac{1 - F(t)}{\mu} = \psi_A(t)$$

and the age of the current process has the same distribution as the entire lifespan of the process.

6.2.3. *distribution of remaining life.* The remaining life or *residual life* of the process at time  $t$  is simply how long we have to wait for the next renewal. It is called  $B_t$ . It is a little more complicated to analyze.

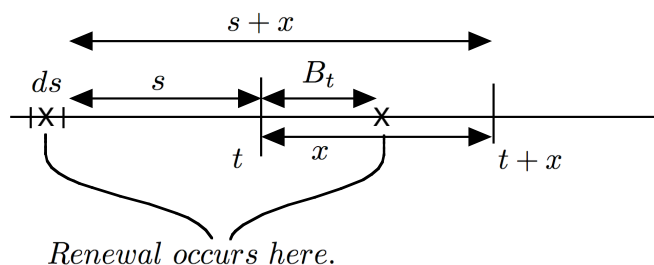


FIGURE 3. The residual life  $B_t$  seems to depend on the time of the last renewal.

We want to calculate the distribution function  $\Psi_B(x) = \mathbb{P}(B_t \leq x)$ . In order for this even to occur, we first need the last renewal to occur in some interval  $ds$  before  $t$ . This has probability  $ds/\mu$ . Then we need the event not to occur again during the time interval  $s$  before time  $t$  but we need it to occur sometime in the time interval  $x$  after time  $t$ . This occurs with probability  $F(s + x) - F(s)$ . But  $s \geq 0$  is arbitrary. So we sum over all  $s$  and get:

$$\Psi_B(x) = \int_0^\infty \frac{ds}{\mu} (F(s + x) - F(s))$$

(See the figure.) To get the density function we should differentiate with respect to  $x$ :

$$\psi_B(x) = \Psi'_B(x) = \int_0^\infty \frac{ds}{\mu} f(s + x)$$

If we substitute  $t = s + x, dt = ds$  we get:

$$\psi_B(x) = \int_x^\infty \frac{dt}{\mu} f(t) = \frac{1 - F(x)}{\mu} = \psi_A(x)$$

In other words, the current age  $A_t$  and the residual life  $B_t$  have the same probability distribution.

6.2.4. *relativity argument.* The symmetry between past and future was the point which I wanted to explain using “relativity.” Instead of having a fixed time  $t$  and looking at the renewal process as occurring at random times, you could think of the renewals as fixed and pick your current time  $t$  at random. If you pick  $t$  in some renewal period of length  $C$  then the point of time that you choose is uniformly distributed (has equal probability of being at any point in the interval). In particular, the left and right parts have the same distribution.

The sum  $C_t := A_t + B_t$  is equal to the total duration of the current process. To find the distribution function for  $C_t$  you can use this relativity argument. Assume that renewal has occurred a very large number of times, say  $N$ . By the law of large number, the total amount of time this takes is very close to  $N\mu$ . Of these  $N$  renewals,  $f(x)dx$  represents the proportion of renewal period of duration  $x$  to  $x + dx$ . So,  $Nf(x)dx$  is the number of times this occurs. Since the renewal periods all have the same length, the total length of time for all of these renewal periods is just the product  $xNf(x)dx$ . If you pick a time at random then the probability that the time you pick will be in one of these intervals is

$$\mathbb{P}(x < C_t \leq x + dx) = \frac{xNf(x)dx}{N\mu} = \frac{xf(x)}{\mu}dx$$

Therefore, the density function for  $C_t$  is  $\psi_C(x) = xf(x)/\mu$ .

For example, for the exponential distribution with rate  $\lambda$ , we get:

$$\psi_C(x) = \frac{xf(x)}{\mu} = \frac{x\lambda e^{-\lambda x}}{1/\lambda} = \lambda^2 x e^{-\lambda x}$$

This is the Gamma-distribution with parameters  $\lambda$  and  $\alpha = 2$ . The reason is that, in this case,  $C_t$  is the sum of two independent exponential variables  $A_t, B_t$  with rate  $\lambda$ .

6.3. **Convolution.** The convolution is used to describe the density function for the sum of independent random variables. It occurs in this chapter because the lifespan of the renewal periods are independent. So, the density function for the  $n$ -th renewal is given by a convolution.

6.3.1. *density of  $X + Y$ .* Suppose that  $X, Y$  are independent random variables with density functions  $f(x), g(y)$  respectively. Then we discussed in class that there are two ways to find the density  $h(z)$  of  $Z = X + Y$ . The first method is intuitive and the second is rigorous.

*Method 1.* I assumed that  $X, Y$  are  $\geq 0$ . But this assumption was not necessary. It just made it easier to talk about.

$h(z)dz$  is the probability that  $X + Y$  will lie in the interval  $[z, z + dz]$ . But in order for this to happen we first need  $X$  to lie in some interval  $[x, x + dx]$  where  $0 \leq x \leq z$ . This occurs with probability  $f(x)dx$ . Then we need  $Y$  to be in the interval  $[z - x, z - x + dz]$ . This occurs with probability  $g(z - x)dz$ . So,

$$h(z)dz = \int_0^z f(x)g(z - x) dx dz$$

Divide by  $dz$  to get

$$h(z) = \int_0^z f(x)g(z - x) dx$$

This is the convolution:

$$h = f * g$$

*Method 2.* Suppose that the distribution functions of  $X, Y, Z$  are  $F, G, H$ . Then

$$H(z) = \mathbb{P}(X + Y \leq z) = \int_{-\infty}^{\infty} G(z - x)f(x)dx$$

Differentiate both sides to get

$$h(z) = \int_{-\infty}^{\infty} g(z - x)f(x)dx$$

6.3.2.  $\Gamma$  *distribution.* Suppose that you have a Poisson process with rate  $\lambda$ . Let  $T$  be the length of time you have to wait for the  $\alpha$ th occurrence of the event. Then  $T$  has a  $\Gamma$  distribution with parameters  $\lambda$  and  $\alpha$ . Since the expected value of the waiting time for the Poisson event is  $1/\lambda$  the expected value of  $T$  must be  $\mathbb{E}(T) = \alpha/\lambda$ .

To get the density function of  $T$  we take the convolution of  $\alpha$  exponential densities:

$$f = \underbrace{\phi * \phi * \phi * \dots * \phi}_{\alpha}$$

For example when  $\alpha = 2$  we get:

$$\begin{aligned} f(t) &= \int_0^t \phi(x)\phi(t - x) dx = \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} dx \\ &= \int_0^t \lambda^2 e^{-\lambda t} dx = \lambda^2 t e^{-\lambda t} \end{aligned}$$

In general you get:

$$f(t) = \frac{1}{(\alpha - 1)!} \lambda^\alpha t^{\alpha-1} e^{-\lambda t}$$

if  $\alpha$  is an integer and for any  $\alpha$ :

$$f(t) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha t^{\alpha-1} e^{-\lambda t}$$

where  $\Gamma(\alpha)$  is what it has to be when  $\lambda = 1$ :

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

One example of the  $\Gamma$ -distribution is  $\chi_r^2$ , the chi-squared distribution with  $r$  degrees of freedom. This is the  $\Gamma$ -distribution with  $\lambda = 1/2$  and  $\alpha = r/2$ .

**6.4. M/G/1-queueing.** In this model, we have people lining up in a queue and one server taking care of these people one at a time. Let's assume the server is a machine.

In the notation “ $M/G/1$ ” the “1” stands for the number of servers. The “ $M$ ” means that the “customers” are entering the queue according to a Poisson process with some fixed rate  $\lambda$ . The “ $G$ ” means that the servers does its job according to some fixed probability distribution which is “general.” i.e., it could be anything. This is a renewal process where “renewal” occurs at the moment the queue is empty. At that time, the system is back in its original state with no memory of what happened.

$X_n = \#$  people who enter the line during the  $n$ -th service period.

$U_n =$  length of time to serve the  $n$ -th person.

So,  $\mathbb{E}(X_n) = \lambda\mu$  where  $\mu = \mathbb{E}(U_n)$ . We need to assume that  $\lambda\mu < 1$ . Otherwise, the line gets longer and long.

$Y_n = \#$  people in queue right after the  $n$ -th person has been served.

Then

$$Y_{n+1} - Y_n = X_{n+1} - 1$$

because  $X_{n+1}$  is the number of people who enter the line and one person leaves. (Let  $Y_0 = 1$  so that the equation also holds for  $n = 0$ .)

**6.4.1. stopping time.** *Busy period* is when the queue and server are active. *Rest periods* are when there is noone in the line. The queue will alternate between busy periods and rest periods. Define the *stopping*

time  $\tau$  to be the number of people served during the first busy period. Then the first busy time (duration of the 1st busy period) is

$$S_1 = U_1 + U_2 + \cdots + U_\tau$$

To find a formula for  $\tau$  we used exercise 5.16 on p.128:

- (a)  $M_n = X_1 + X_2 + \cdots + X_n - n\mathbb{E}(X)$  is a uniformly integrable martingale.
- (b)  $M_0 = 0$
- (c)  $OST \Rightarrow \mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0$ . This gives us:
- (d) (*Wald's equation*)

$$\mathbb{E}(X_1 + \cdots + X_\tau) = \mathbb{E}(\tau)\mathbb{E}(X)$$

But the sum of the numbers  $X_n$  gives the total number of people who entered the line after the first person. So:

$$X_1 + \cdots + X_\tau = \tau - 1$$

Put this into Wald's equation and we get:

$$\mathbb{E}(\tau) - 1 = \mathbb{E}(\tau)\mathbb{E}(X) = \mathbb{E}(\tau)\lambda\mu$$

where  $\mu = \mathbb{E}(U)$ . Solve for  $\mathbb{E}(\tau)$  to get

$$\mathbb{E}(\tau) = \frac{1}{1 - \lambda\mu}$$

6.4.2. *equilibrium distribution.* We want to know about the equilibrium distribution of the numbers  $Y_n$ . The stopping time  $\tau$  is the smallest number so that  $Y_\tau = 0$ . This means that  $\tau$  is the time it takes for  $Y_n$  to go from state  $Y_0 = 1$  to state  $Y_\tau = 0$ . So  $\tau + 1$  is the number of steps to go from 0 to 0. (In one step it goes to 1.) Therefore, in the equilibrium distribution  $\pi$  of the Markov chain  $Y_n$  we have

$$\mathbb{E}(\tau) + 1 = \frac{1}{\pi_0}$$

or

$$\pi_0 = \frac{1}{\mathbb{E}(t) + 1} = \frac{1 - \lambda\mu}{2 - \lambda\mu}$$