

0. DIFFERENTIAL AND DIFFERENCE EQUATIONS

We have two days to go over the basics of linear differential equations. Differential equations is a one semester course and we don't have time to cover it in detail. However, to do Markov chains you just need to understand how first order linear differential equations work.

0.1. Linear differential equations in one variable. In the first lecture, I discussed linear differential equations (Diffeq's) in one variable of arbitrary order. I presented the problem and the complete solution but without proof. These missing proofs I appended at the end so that these notes will faithfully represent the style and content of the lectures.

The problem in degree $d = 2$ is to find a function $y = f(t)$ so that:

$$(0.1) \quad y'' + ay' + by + c = 0$$

where a, b, c are constants. (The *degree*, or *order* is the number of times that the variables are differentiated. In this case the degree is 2 since we have y'' .)

0.1.1. *particular solution.* A (one) solution $y = f_0(t)$ of this equation is called a *particular solution*. It is really easy to find:

$$y = f_0(t) = \frac{-c}{b}.$$

This is a constant function. It's derivative (and higher derivatives) are zero: $y' = y'' = 0$. So, when you plug it into Equation (0.1) you get

$$0 + 0 + by + c = 0 \Rightarrow y = -c/b.$$

If $b = 0$ then the answer is

$$y = f_0(t) = \frac{-c}{a}t.$$

This is also easy to see: $y' = -c/a$ and $y'' = 0$. So,

$$y'' + ay' + by + c = 0 + a(-c/a) + 0 + c = -c + c = 0.$$

Now, suppose you have another solution $y = f(t)$.

0.1.2. *homogeneous equation.*

Lemma 0.1. *If $f_0(t), f(t)$ are two solutions of the differential equation then the difference*

$$y = f(t) - f_0(t)$$

is a solution of the homogeneous equation

$$y'' + ay' + by = 0.$$

(This is the original equation minus the constant term c .)

This lemma implies that the solutions of the Diffeq are given by:

$$f(t) = f_0(t) + (\text{all solutions of the homogeneous equation})$$

The solutions of homogeneous equation have good theoretical properties:

Lemma 0.2. *If $f_1(t)$ is a solution of the homogeneous equation then so is $\alpha f_1(t)$ for any (constant) scalar α .*

Lemma 0.3. *If f_1, f_2 are two solutions of the homogeneous equation then so is $y = f_1 + f_2$.*

These two lemmas imply that we have a vector space:

Theorem 0.4. *The set of solutions of the homogeneous equation form a vector space. The dimension of this space is equal to the degree d of the differential equation.*

As you remember from linear algebra, every vector space has a basis: $f_1(t), f_2(t), \dots, f_d(t)$. So, every solution of the homogeneous equation is a linear combination of the basis elements:

$$a_1 f_1 + a_2 f_2 + \dots + a_d f_d.$$

This means that the solutions of the original Diffeq are given by:

$$f(t) = f_0(t) + a_1 f_1(t) + a_2 f_2(t) + \dots + a_d f_d(t)$$

where a_1, \dots, a_d are arbitrary scalars.

Thus, we need d linearly independent solutions of the homogeneous differential equation.

0.1.3. *solutions for homogeneous equation of degree one.* I first did the case of degree one. This is an equation of the form:

$$y' + ay = 0$$

or:

$$y' = -ay.$$

I.e., y is decreasing at a rate proportional to its size. This is the equation of exponential decay which you learn in calculus. The solution is

$$y = y_0 e^{-at}.$$

0.1.4. *solutions in higher order.* To explain the idea, I gave a specific example:

$$y'' - 5y' + 6y + 1 = 0.$$

The particular solution is $f_0(t) = -1/6$ and the homogeneous equation is:

$$y'' - 5y' + 6y = 0.$$

I rewrote this using the *differential operator* $\mathcal{D} = \frac{d}{dt}$:

$$\mathcal{D}^2 y - 5\mathcal{D}y + 6y = 0$$

or

$$(\mathcal{D}^2 - 5\mathcal{D} + 6)y = 0.$$

Now we can factor the operator (the thing that is operating on y):

$$(\mathcal{D} - 2) \underbrace{(\mathcal{D} - 3)}_{=0} y = 0.$$

To solve this, we look at the part $(\mathcal{D} - 3)y$. If this is zero then the whole thing is zero. But the equation

$$(\mathcal{D} - 3)y = 0$$

is the first order equation

$$y' - 3y = 0.$$

The solution is $y = C_1 e^{3t}$ for some constant C_1 . This gives one basis element $f_1(t) = e^{3t}$. To get the other one, we go back and rewrite the differential equation as:

$$(\mathcal{D} - 3)(\mathcal{D} - 2)y = 0.$$

Then we get the solution $y = C_2 e^{2t}$ and $f_2(t) = e^{2t}$. So, the general solution of the Diffeq is

$$f(t) = -\frac{1}{6} + C_1 e^{3t} + C_2 e^{2t}.$$

Theorem 0.5. *If we have a linear differential equation in one variable of order d given by a polynomial in \mathcal{D} of degree d with d distinct roots $\lambda_1, \lambda_2, \dots, \lambda_d$ then the functions $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_d t}$ form a basis for the vector space of all solutions of the associated homogeneous equation.*

In the example, $d = 2$, the polynomial is $\mathcal{D}^2 - 5\mathcal{D} + 6$ which has roots $\lambda_1 = 3, \lambda_2 = 2$. The analysis of the general case is very similar.

There are two points which I expanded on in order to give a complete description of the answer:

- complex roots,
- multiple roots.

0.1.5. *complex roots.* What happens in the case when λ is a complex number? I gave an example to start:

$$y'' + 2y' + 5y = 0.$$

In terms of differential operators this is:

$$(\mathcal{D}^2 + 2\mathcal{D} + 5)y = 0.$$

The roots of this polynomial are:

$$\lambda_{\pm} = -1 \pm 2i, \quad i = \sqrt{-1}.$$

So, a basis for the solution space is $e^{\lambda_+ t}, e^{\lambda_- t}$. But what are these functions? Here I switched to letters: $\lambda_{\pm} = a \pm bi$ where $a = -1, b = 2$.

$$e^{\lambda_+ t} = e^{at} e^{bit} = e^{at} (\cos bt + i \sin bt)$$

$$e^{\lambda_- t} = e^{at} e^{-bit} = e^{at} (\cos bt - i \sin bt)$$

If we add these and divide by 2 or subtract and divide by $2i$ we get two other basic solutions of the homogeneous equation:

$$f_1(t) = e^{at} \cos bt$$

$$f_2(t) = e^{at} \sin bt$$

In our particular example, we have

$$f_1(t) = e^{-t} \cos 2t$$

$$f_2(t) = e^{-t} \sin 2t.$$

These two functions form the *real* basis for the 2-dimensional vector space of all solutions of the second order homogeneous diffeq.

0.1.6. *multiple roots.* Suppose that the polynomial has multiple roots. For example, suppose the equation is:

$$y'' + 4y' + 4y = 0$$

$$(\mathcal{D}^2 + 4\mathcal{D} + 4)y = 0.$$

This factors as:

$$(\mathcal{D} + 2)^2 y = 0.$$

The roots are $\lambda = -2, -2$. I.e., -2 is a *double root*.

We know that one of the solutions is $f_1(t) = e^{-2t}$. There must be one more. We cannot take the same function twice. The general answer is

$$f_2(t) = te^{\lambda t} = e^{-2t}.$$

If λ is a triple root we get $f_3 = t^2 e^{\lambda t}$ and so on.

The derivation, which I did not give in class, is easy:

$$\mathcal{D}(te^{\lambda t}) = e^{\lambda t} + t\lambda e^{\lambda t}.$$

So,

$$\begin{aligned}(\mathcal{D} - \lambda)te^{\lambda t} &= e^{\lambda t} \\ (\mathcal{D} - \lambda)(\mathcal{D} - \lambda)te^{\lambda t} &= (\mathcal{D} - \lambda)e^{\lambda t} = 0.\end{aligned}$$

This gives the complete description of the solution of a linear differential equation in one variable.

0.1.7. *proofs.* Students should feel free to skip this subsection and go on to the section on epidemic modeling. This is only for those who want to see all the details.

Going back to our original equation

$$y'' + ay' + by + c = 0$$

we rewrite this as

$$\begin{aligned}(\mathcal{D}^2 + a\mathcal{D} + b)y + c &= 0 \\ \varphi(y) &= -c\end{aligned}$$

where

$$\varphi = \mathcal{D}^2 + a\mathcal{D} + b.$$

The point is that φ is a *linear operator*, i.e.,

$$\varphi(\alpha f_1 + \beta f_2) = \alpha\varphi f_1 + \beta\varphi f_2$$

if α, β are constants. If $\varphi f_1 = \varphi f_2 = 0$ then this equation implies that $\varphi(\alpha f_1 + \beta f_2) = 0$. This proves Lemma 0.2, Lemma 0.3 and the first sentence in Theorem 0.4. It remains to prove Lemma 0.1 and the rest of Theorem 0.4.

Proof of Lemma 0.1. If f_0, f are solutions of the original Diffeq then $\varphi(f_0) = -c$ and $\varphi(f) = -c$. So,

$$\varphi(f - f_0) = \varphi f - \varphi f_0 = -c + c = 0.$$

I.e., $f - f_0$ is a solution of the homogeneous equation. \square

Proof of Theorem 0.4. The proof is by induction on d . If $d = 1$ then you learned, or should have learned, the following in calculus.

$$\begin{aligned}(\mathcal{D} - \lambda)y &= 0 \\ \mathcal{D}y &= \lambda y \\ \frac{dy}{dt} &= \lambda y.\end{aligned}$$

Separate variables:

$$\frac{dy}{y} = \lambda dt,$$

then integrate:

$$\int \frac{dy}{y} = \int \lambda dt.$$

The solution is:

$$\begin{aligned} \ln |y| &= \lambda t + C \\ y &= \pm e^C e^{\lambda t} = y_0 e^{\lambda t} \end{aligned}$$

where $y_0 = \pm e^C$ is constant, i.e., a scalar. So, $e^{\lambda t}$ is a basis. The vector space has dimension $d = 1$ as claimed. This case can be written as follows.

Lemma 0.6. *The kernel $\ker(\mathcal{D} - \lambda)$ of the linear operator $\mathcal{D} - \lambda$ is one dimensional and is spanned by $e^{\lambda t}$.*

Now suppose the theorem holds for degree $d - 1$. Since any polynomial in \mathcal{D} can be factor as a product of linear factors $\mathcal{D} - \lambda$ for complex numbers λ , we can write:

$$\varphi = \psi(\mathcal{D} - \lambda)$$

where ψ is a polynomial of degree $d - 1$ in \mathcal{D} . By induction on d we know that the solution space of the homogeneous Diffeq $\psi y = 0$ is a vector space of dimension $d - 1$, i.e.,

$$\dim \ker \psi = d - 1.$$

But,

$$f(t) \in \ker \varphi \Leftrightarrow \psi(\mathcal{D} - \lambda)f(t) = 0 \Leftrightarrow (\mathcal{D} - \lambda)f(t) \in \ker \psi.$$

So, $\mathcal{D} - \lambda$ is a linear mapping

$$\mathcal{D} - \lambda = L : \ker \varphi \rightarrow \ker \psi.$$

We just showed that the kernel of $\mathcal{D} - \lambda$ is one-dimensional. So,

$$\begin{aligned} \dim(\ker \varphi) &= \dim(\ker L) + \dim(\text{im } L) \\ &\leq 1 + \dim \ker \psi = 1 + (d - 1) = d. \end{aligned}$$

This means that there are *at most* d linearly independent solutions of a homogeneous linear Diffeq of order d . But, in the discussion above we found d linearly independent solutions. So, we know that

$$\dim \ker \varphi \geq d.$$

So, we must have

$$\dim \ker \varphi = d$$

as claimed. □