

0.2. Kermack-McKendrick. This is from the book *Epidemic Modelling, An Introduction*, D.J. Daley & J.Gani, Cambridge University Press. Kermack-McKendrick is the most common model for the general epidemic. It is usually more realistic with many subpopulations with different characteristics. But we are only interested in the concept, not in an accurate model. So, I made two simplifying assumptions:

- The population is homogeneous
- No births or deaths by other means

Since there are no births, the size of the population N is constant.

This model is similar to the Markov processes which we will study starting next week: There are “states” and people move from one state to another according to certain rules. In a Markov process, the movement is random. Here it is deterministic.

In this model there are three states:

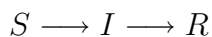
- S:** = susceptible
- I:** = infected
- R:** = removed (immune)

Let $x = \#S, y = \#I, z = \#R$. So

$$N = x + y + z.$$

I assume that $z_0 = 0$ (If there are any “removed” people at $t = 0$ we ignore them.)

As time passes, susceptible people become infected and infected “recover” and become immune. So the size of S decreases and the size of R increases. People move as shown by the arrows:



0.2.1. recovery rate. The infected recover at an exponential rate. We assume that the infection has a half-life, say one week, and half of the infected will recover in that time and $3/4$ will recover in two weeks, etc. So, the number of infected tends to decrease at the rate proportional to its size. However, there are also newly infected which keep appearing. So, this recovery process only describes the flow $I \rightarrow R$. The equation is:

$$\boxed{\frac{dz}{dt} = \gamma y, \quad \gamma > 0.}$$

The rate of change of y is equal to the rate of infection minus the rate of recovery.

0.2.2. *infection rate.* The infection rate is given by the *Law of mass action* which says:

The rate of interaction between two different subsets of the population is proportional to the product of the number of elements in each subset.

So,

$$\boxed{\frac{dx}{dt} = -\beta xy, \quad \beta > 0.}$$

To solve these equations, we divide them:

$$\frac{dx}{dz} = \frac{dx/dt}{dz/dt} = \frac{-\beta xy}{\gamma y} = \frac{-\beta x}{\gamma}$$

This is a linear differential equation with solution

$$x = x_0 \exp\left(\frac{-\beta}{\gamma} z\right) = x_0 e^{-z/\rho}$$

where $\rho := \gamma/\beta$ is called the *threshold* population size. This is an exponential decay equation. It says that the size of the susceptible population is decreasing at an alarming rate. Bad news!

However, something happens before we all die: the number of infected y goes to zero and the infection stops!

Since $N = x + y + z$ is fixed we can find y as a function of z :

$$y = N - x - z = N - x_0 e^{-z/\rho} - z$$

Differentiating gives:

$$\frac{dy}{dz} = \frac{x_0}{\rho} e^{-z/\rho} - 1$$

$$\frac{d^2y}{dz^2} = -\frac{x_0}{\rho^2} e^{-z/\rho} < 0$$

So, the function is concave down with initial slope

$$\boxed{\frac{dy}{dz} = \frac{x_0}{\rho} - 1.}$$

I graphed these functions for different values of the parameters to show you what this means.

0.2.3. *Case 1: $x_0 > \rho$.* When the initial susceptible population size is greater than the threshold ρ , the infected population increases at the beginning. This is because

$$x_0 > \rho \quad \Rightarrow \quad \frac{x_0}{\rho} > 1 \quad \Rightarrow \quad \frac{dy}{dz} = \frac{x_0}{\rho} - 1 > 0.$$

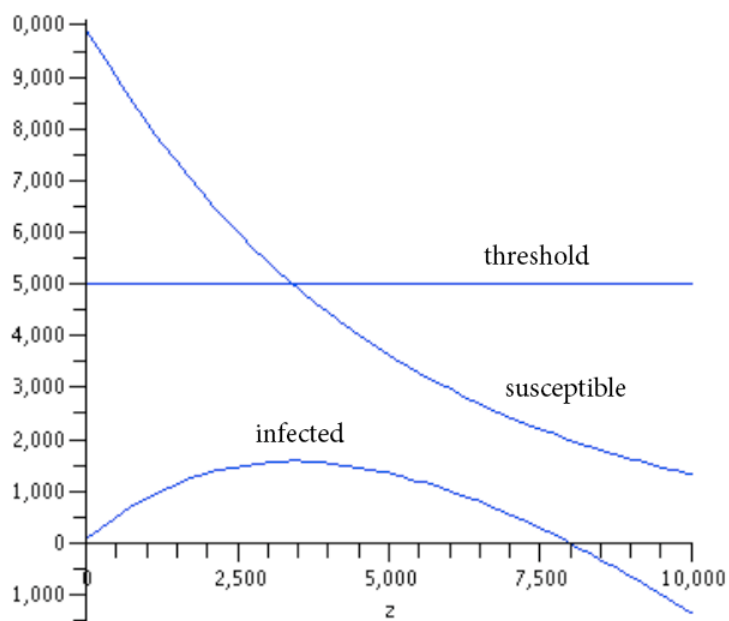
However, it eventually comes back down, although this may not be obvious. Here is the plot in the case when

$$N = 10,000$$

$$x_0 = 9,900$$

$$y_0 = 100$$

$$\rho = 5,000.$$



The plot shows x, y, ρ as a function of z . (ρ is constant.)

Note that there are approximately 2,000 uninfected at the end of the epidemic. (The infected line crosses the z axis at $z = 8,000$ and $x = N - z = 2,000$ is the final value.)

In fact, this model predicts that there will always be survivors of any epidemic. I.e., there will always be people who never get infected.

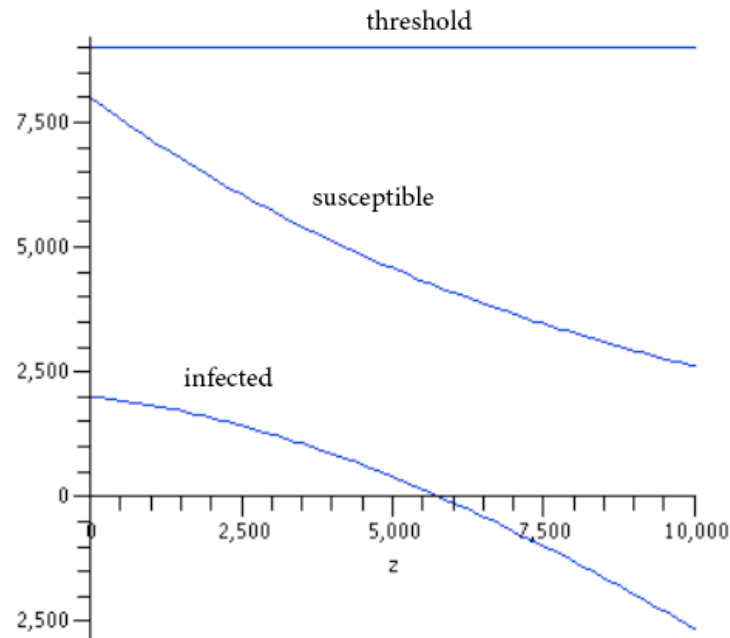
0.2.4. *case 2: $x_0 < \rho$.* If the initial susceptible population size is less than the threshold ρ , the infected population is decreasing at the beginning. Since the infected curve is concave down, it decreases even faster as z increases. Here is the plot in the case

$$N = 10,000$$

$$x_0 = 8,000$$

$$y_0 = 2,000$$

$$\rho = 9,000.$$



Here, almost half of the population survives the epidemic.

Another way to look at it is that this is the tail end of the epidemic. The worst is over. In case 1 we saw the beginning of the epidemic.

Exercise 0.7. Prove that the highest point in the infected curve occurs when the susceptible curve crosses the threshold.