0.3. **systems of first order equations.**

(1) Linear differential equations can be reduced to first order.
(2) First order equations are matrix equations.
(3) Exponential of a matrix.
(4) Eigenvalues and eigenvectors.
(5) Diagonalization of a matrix.
(6) Solution.

0.3.1. *reduction to first order.*

Theorem 0.8. *Any nth order linear differential equation in m variables can be reduced to a first order linear diffeq in nm variables.*

Here is an example with $n = m = 2$.

$$x'' = 2x' + 3y' + 6x + 7$$
$$y'' = x' + 9y - 5.$$

(Differentiation is with respect to time t.) The theorem says you need $nm = 4$ variables, i.e., 2 more. Call them v, w and let

$$v = x', \quad w = y'$$

Then we have four 1st order linear diffeq's in 4 variables:

$$v' = 2v + 3w + 6x + 7$$
$$w' = v + 9y - 5$$
$$x' = v$$
$$y' = w.$$

You always get derivatives of the variable on the left and linear combinations of the variables on the right.

This example is too complicated. So, I started over with a simpler example.

Example 0.9. Solve the following 2nd order differential equation:

$$y'' + y' - 6y = 0$$

with initial conditions:

$$y_0 = 0, \quad y_0' = 5.$$

(Initial position at the origin with initial velocity 5.)

Following the procedure, you introduce another variable $z := y'$. Then the 1st order equations are:

$$y' = z$$
$$z' = 6y - z$$

0.3.2. *matrix form.* In matrix form this is:

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

Which we can write as

$$\boxed{\frac{d}{dt}Y = AY}$$

where

$$Y = \begin{pmatrix} y \\ z \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}.$$

0.3.3. *exponential of a matrix.* The solution of this equations is

$$\boxed{Y = e^{tA}Y_0}$$

where $Y_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ and the matrix exponential e^{tA} is defined by:

$$e^{tA} := I_2 + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{t^kA^k}{k!}$$

To prove that this is the solution, you differentiate the sum term by term:

$$\frac{d}{dt}e^{tA} = \frac{d}{dt}\sum \frac{t^kA^k}{k!} = \sum \frac{d}{dt}\frac{t^kA^k}{k!} = \sum \frac{kt^{k-1}A^k}{k!}$$

which simplifies to

$$\sum \frac{t^{k-1}A^k}{(k-1)!} = A\sum \frac{t^{k-1}A^{k-1}}{(k-1)!} = Ae^{tA}.$$

So,

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

0.3.4. *eigenvalues and eigenvectors.* In order to compute e^{tA} we need the eigenvalues and eigenvectors of the matrix A. These are given by the equation

$$\boxed{AX = \lambda X = X(\lambda)}.$$

In this example we have:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{X} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \underbrace{2}_{\lambda}\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}(2)$$

(If λ is a scalar, it belongs on the left. If (λ) is a 1×1 matrix, it must be on the right by the rules of matrix multiplication.)

$\lambda_1 = 2$ is an eigenvalue of our matrix A with eigenvector $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The other eigenvalue is $\lambda_2 = -3$ with eigenvector $X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} (-3)$$

0.3.5. *diagonalization.* Put these together to get:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}}_{Q=(X_1X_2)} = \begin{pmatrix} 2 & -3 \\ 4 & 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}}_{Q=(X_1X_2)} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}}_{D}.$$

Or:

$$AQ = QD$$

where $Q = (X_1X_2)$ is the matrix whose columns are the eigenvectors of A and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is the diagonal matrix whose diagonal entries are the eigenvalues. The equation $AQ = QD$ should be rewritten as:

$$\boxed{A = QDQ^{-1}.}$$

This is called the *diagonalization* of A.

0.3.6. *powers of A.* After we successfully diagonalized A, we can raise it to a power: First,

$$A^2 = QD\underbrace{Q^{-1}Q}_{I_2}DQ^{-1} = QDI_2DQ^{-1} = QD^2Q^{-1}.$$

By induction we get:

$$A^k = QD^kQ^{-1}.$$

Divide by $k!$ and sum over k to get:

$$e^{tA} = Qe^{tD}Q^{-1} = Q\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} Q^{-1}$$

0.3.7. *solution.* The inverse of

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$$

is given by

$$Q^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-5} \begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}.$$

So,

$$Y = e^{tA} Y_0 = Q e^{tD} Q^{-1} Y_0.$$

Putting in all the numbers, including $y_0 = 0, z_0 = y_0' = 5$, we get:

$$\begin{aligned}
\begin{pmatrix} y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} \\ -e^{-3t} \end{pmatrix} = \begin{pmatrix} e^{2t} - e^{-3t} \\ 2e^{2t} + 3e^{-3t} \end{pmatrix}.
\end{aligned}$$

In other words,

$$y = e^{2t} - e^{-3t}$$
$$z = y' = 2e^{2t} + 3e^{-3t}.$$

This algorithm works if the eigenvalues of A are distinct.

0.3.8. *review of linear algebra.* The eigenvalues of a square matrix are defined to be the solutions of the equation

$$\det(A - \lambda I) = 0$$

but there is a trick to use for 2×2 matrices. The determinant of a matrix is always the product of its eigenvalues:

$$\det A = \lambda_1 \lambda_2 = -6$$

The trace (sum of diagonal entries) is equal to the sum of the eigenvalues:

$$tr A = \lambda_1 + \lambda_2 = -1$$

So, $\lambda_1 = 2, \lambda_2 = -3$. The eigenvectors are the solutions of the linear equations

$$AX = \lambda X$$

or

$$(A - \lambda I_2)X = 0.$$