

0.4. Linear difference equations.

0.4.1. *discrete time.* To make the transition

differential equations \Rightarrow Markov chains

we convert to *discrete time* and *row vectors*¹.

$$t \in \mathbb{R}_+(\text{continuous time}) \Rightarrow n \in \mathbb{Z}_+(\text{discrete time})$$

$$Y : \text{column vector} \Rightarrow X : \text{row vector.}$$

The solution we get is

$$X = X_0 e^{tA} = X_0 P^n$$

where $P = e^A$ and $n = t$ (because it is an integer). The exponential matrix function is replaced with positive integer powers of a matrix P .

0.4.2. *one variable, second order.* Discrete time equations have a one variable higher order form and a multivariable first order matrix form. We will start with a one variable second order equation in discrete time.

The problem is to find a sequence of numbers $f(n)$ where n ranges over all the integers from K to N ($K \leq n \leq N$) so that

$$(0.2) \quad f(n) = af(n-1) + bf(n+1)$$

The theory is the same as for DiffEq's. This is homogeneous, linear, second order. So, it has two linearly independent solutions.

The continuous solutions were $e^{t\lambda}$. In discrete time this is

$$e^{t\lambda} = c^n$$

where $c = e^\lambda$ and $n = t$ is an integer. So, $f(n) = c^n$ where you have to solve for c :

$$c^n = ac^{n-1} + bc^{n+1}$$

$$(0.3) \quad bc^2 - c + a = 0$$

$$c = \frac{1 \pm \sqrt{1 - 4ab}}{2b}$$

There were two cases.

Case 1: ($4ab \neq 1$) When the quadratic equation (0.3) has two roots c_1, c_2 then the linear combinations of c_1^n and c_2^n give all the solutions of the homogeneous linear recursion (0.2).

Case 2: ($4ab = 1$) In this case there is only one root $c = \frac{1}{2b}$ and the two independent solutions are $f(n) = c^n$ and nc^n . The second solution is the discrete form of:

$$te^{t\lambda} = nc^n$$

¹I will show you later the rigorous proof that transposing the matrix changes deterministic differential equations into random stochastic processes!!

where $t = n, c = e^\lambda$.

0.4.3. *examples.* An example of Case 1:

Example 0.10. (Fibonacci numbers) These are given by $f(0) = 1, f(1) = 1$ and $f(n+1) = f(n) + f(n-1)$ or:

$$f(n) = f(n+1) - f(n-1)$$

This is $a = -1, b = 1$. The roots of the quadratic equation are $c = \frac{1 \pm \sqrt{5}}{2}$. So,

$$f(n) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

This is a rational number since it is Galois invariant (does not change if you switch the sign of $\sqrt{5}$). However, it is not clear from the formula why it is an integer.

Here is an example of Case 2:

Example 0.11. Solve the linear difference equation

$$f(n) = \frac{f(n-1) + f(n+1)}{2}$$

with initial conditions $f(0) = 1, f(1) = 2$. In this case $a = b = \frac{1}{2}$. So, $4ab = 1$ and $c = \frac{1}{2b} = 1$. This means the two linearly independent solutions are

$$\begin{aligned} f_1(n) &= c^n = 1 \\ f_2(n) &= nc^n = n. \end{aligned}$$

So,

$$f(n) = xf_1(n) + yf_2(n) = x + ny.$$

Plugging in $n = 0, 1$ we get:

$$\begin{aligned} f(0) &= 1 = x, \\ f(1) &= 2 = x + y. \end{aligned}$$

So, $x = y = 1$ and

$$f(n) = 1 + n.$$