

#### 0.4. Linear difference equations.

0.4.1. *discrete time.* To make the transition

differential equations  $\Rightarrow$  Markov chains

we convert to *discrete time* and *row vectors*<sup>1</sup>.

$$t \in \mathbb{R}_+(\text{continuous time}) \Rightarrow n \in \mathbb{Z}_+(\text{discrete time})$$

$$Y : \text{column vector} \Rightarrow X : \text{row vector.}$$

The solution we get is

$$X = X_0 e^{tA} = X_0 P^n$$

where  $P = e^A$  and  $n = t$  (because it is an integer). The exponential matrix function is replaced with positive integer powers of a matrix  $P$ .

0.4.2. *one variable, second order.* Discrete time equations have a one variable higher order form and a multivariable first order matrix form. We will start with a one variable second order equation in discrete time.

The problem is to find a sequence of numbers  $f(n)$  where  $n$  ranges over all the integers from  $K$  to  $N$  ( $K \leq n \leq N$ ) so that

$$(0.2) \quad f(n) = af(n-1) + bf(n+1)$$

The theory is the same as for DiffEq's. This is homogeneous, linear, second order. So, it has two linearly independent solutions.

The continuous solutions were  $e^{t\lambda}$ . In discrete time this is

$$e^{t\lambda} = c^n$$

where  $c = e^\lambda$  and  $n = t$  is an integer. So,  $f(n) = c^n$  where you have to solve for  $c$ :

$$c^n = ac^{n-1} + bc^{n+1}$$

$$(0.3) \quad bc^2 - c + a = 0$$

$$c = \frac{1 \pm \sqrt{1 - 4ab}}{2b}$$

There were two cases.

Case 1: ( $4ab \neq 1$ ) When the quadratic equation (0.3) has two roots  $c_1, c_2$  then the linear combinations of  $c_1^n$  and  $c_2^n$  give all the solutions of the homogeneous linear recursion (0.2).

Case 2: ( $4ab = 1$ ) In this case there is only one root  $c = \frac{1}{2b}$  and the two independent solutions are  $f(n) = c^n$  and  $nc^n$ . The second solution is the discrete form of:

$$te^{t\lambda} = nc^n$$

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<sup>1</sup>I will show you later the rigorous proof that transposing the matrix changes deterministic differential equations into random stochastic processes!!

where  $t = n, c = e^\lambda$ .

0.4.3. *examples.* An example of Case 1:

**Example 0.10.** (Fibonacci numbers) These are given by  $f(0) = 1, f(1) = 1$  and  $f(n+1) = f(n) + f(n-1)$  or:

$$f(n) = f(n+1) - f(n-1)$$

This is  $a = -1, b = 1$ . The roots of the quadratic equation are  $c = \frac{1 \pm \sqrt{5}}{2}$ . So,

$$f(n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

This is a rational number since it is Galois invariant (does not change if you switch the sign of  $\sqrt{5}$ ). However, it is not clear from the formula why it is an integer.

Here is an example of Case 2:

**Example 0.11.** Solve the linear difference equation

$$f(n) = \frac{f(n-1) + f(n+1)}{2}$$

with initial conditions  $f(0) = 1, f(1) = 2$ . In this case  $a = b = \frac{1}{2}$ . So,  $4ab = 1$  and  $c = \frac{1}{2b} = 1$ . This means the two linearly independent solutions are

$$\begin{aligned} f_1(n) &= c^n = 1 \\ f_2(n) &= nc^n = n. \end{aligned}$$

So,

$$f(n) = xf_1(n) + yf_2(n) = x + ny.$$

Plugging in  $n = 0, 1$  we get:

$$\begin{aligned} f(0) &= 1 = x, \\ f(1) &= 2 = x + y. \end{aligned}$$

So,  $x = y = 1$  and

$$f(n) = 1 + n.$$