

MATH 56A SPRING 2008  
STOCHASTIC PROCESSES

KIYOSHI IGUSA

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## 2. COUNTABLE MARKOV CHAINS

We will now talk about *countably infinite* Markov chains. The concept is the same but having an infinite number of states makes several qualitative differences.

**2.1. Concept and examples.** We will study Markov chains with countably infinite number of states. For example, we could take the nonnegative integers:

$$S = \{0, 1, 2, 3, 4, \dots\}$$

or all integers:

$$S = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We will also study the set of integer points in  $n$ -dimensional space:

$$S = \mathbb{Z}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \text{each } x_i \in \mathbb{Z}\}.$$

This is called the *integer lattice* in  $\mathbb{R}^n$ .

The *transition matrix*  $P$  is now infinite:

$$P = (p_{ij}) = (p(i, j)).$$

The numbers still mean the same thing:

$$p_{ij} = p(i, j) = \mathbb{P}(X_{n+1} = j \mid X_n = i).$$

And they still satisfy the condition that  $0 \leq p_{ij} \leq 1$  and the rows add up to one. But now this means that we have an infinite series adding to 1. For example, if  $S = \{0, 1, 2, \dots\}$  then the condition is:

$$p_{i0} + p_{i1} + p_{i2} + p_{i3} + \dots = 1.$$

In the general case, the condition is:

$$\sum_j p_{ij} = 1.$$

**2.1.1. example: random walk on  $\mathbb{Z}$ .** In the first example I only posed the question but didn't answer it. This is the example where  $S = \mathbb{Z}$  and, at each point, there is a probability of  $p$  of going to the right and  $1 - p$  of going to the left:



$$\mathbb{P}(X_{n+1} = x + 1 \mid X_n = x) = p(x, x + 1) = p$$

$$\mathbb{P}(X_{n+1} = x - 1 \mid X_n = x) = p(x, x - 1) = 1 - p$$

The question is: If  $p \geq \frac{1}{2}$  then how certain is it that you will go to  $+\infty$ ? You might want to write the question as:

$$\mathbb{P}(\lim_{n \rightarrow \infty} X_n = \infty) = ?$$

However, this doesn't quite make sense because the limit may not exist. The Markov chain might (and will) oscillate wildly. The correct question is:

$$\mathbb{P}(\limsup X_n = \infty) = ?$$

**Definition 2.1.** For any sequence of real numbers  $x_n$ ,

$$\limsup x_n := \lim_{n \rightarrow \infty} (\sup\{x_n, x_{n+1}, x_{n+2}, \dots\})$$

where “sup” is short for *supremum* which means *maximum* for a finite set and *least upper bound* for an infinite set.

For any sequence of real numbers  $x_0, x_1, \dots$  the supremums

$$\sup\{x_0, x_1, \dots\} \geq \sup\{x_1, x_2, \dots\} \geq \sup\{x_2, x_3, \dots\} \geq \dots$$

form a nonincreasing sequence of real numbers which may all be  $+\infty$ . Therefore, they always have a limit in the set of real numbers union with  $+\infty$ :

$$\limsup x_n \in [-\infty, +\infty].$$

(Any bounded nonincreasing sequence has a limit.)

Similarly, the “liminf” is defined by

$$\liminf x_n = \lim_{n \rightarrow \infty} (\inf\{x_n, x_{n+1}, \dots\}) \in [-\infty, \infty].$$

**2.2. extinction probability.** In the next example,  $S = \{0, 1, 2, 3, \dots\}$  is the population size of some single celled organism. Suppose that, in each generation, every cell either:

- (1) dies with probability  $p_0$  or
- (2) divides into  $k$  cells with probability  $p_k$ .

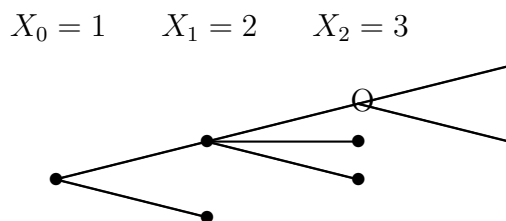
Since these are all the possibilities we have:

$$p_0 + p_1 + p_2 + \dots = 1.$$

We have a countable Markov chain where

$$X_n = \# \text{ cells in generation } n.$$

**2.2.1. branching process.** This is an example of a *branching process*. To explain what this means I drew a picture:



Each cell, for example the cell indicated with a circle, starts a branch which looks just like the whole thing. In other words, we have a “fractal” in which every piece looks like the whole. This is a branching process.

**2.2.2. question.** The question is: What is the *extinction probability*? This is

$$a := \mathbb{P}(\liminf X_n = 0 \mid X_0 = 1).$$

This is the probability of extinction when we start with one cell. But what if we start with more than one?

**Lemma 2.2.**  $\mathbb{P}(\liminf X_n = 0 \mid X_0 = k) = a^k$ .

*Proof.* I drew a picture to indicate that for each of the  $k$  starting cells, there is a probability  $a$  that that cell and all of its descendants will die out. Assuming independence, the probability that this will happen to each of  $k$  cells is  $a^k$  since we multiply probabilities of independent events.  $\square$

With this lemma we get a formula for the extinction probability:

$$\begin{aligned} a &= \mathbb{P}(X_\infty = 0 \mid X_0 = 1) \\ &= \sum_{k=0}^{\infty} \underbrace{\mathbb{P}(X_\infty = 0 \mid X_1 = k)}_{a^k} \underbrace{\mathbb{P}(X_1 = k \mid X_0 = 1)}_{p_k} \\ &= \sum_{k=0}^{\infty} a^k p_k. \end{aligned}$$

This is, by definition, the generating function of the sequence  $p_k$ .

2.2.3. *generating functions.* The definition of the generating function  $\phi(s)$  is

$$\phi(s) := \sum_{k=0}^{\infty} s^k p_k = p_0 + s p_1 + s^2 p_2 + s^3 p_3 + \dots$$

So, the extinction probability  $a$  is the solution of the equation:

$$\boxed{a = \phi(a)}$$

We looked at the general properties of the generating function and fed in the specifics of our example. First,

$$\phi(0) = p_0 > 0.$$

We are going to *assume* that  $p_0 > 0$ . Otherwise, every cell is immortal and there will be no chance of extinction. Next,

$$\phi(1) = \sum p_k = p_0 + p_1 + p_2 + \dots = 1.$$

This is because the probabilities add to 1.

$$\phi'(s) = \sum_{k=1}^{\infty} k s^{k-1} p_k = p_1 + 2s p_2 + 3s^2 p_3 + \dots$$

$$\phi'(1) = \sum k p_k = \mathbb{E}(K) = \mu.$$

This is, by definition, the expected value  $\mu = \mathbb{E}(K)$  of the random variable  $K$  which is the number of offspring of any one cell. It is standard probability notation that the lower case letters are the particular values of upper case letters.

We also need the second derivative:

$$\phi''(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} p_k = 2p_2 + 6s p_3 + 12s^2 p_4 + \dots$$

This function is  $\geq 0$  for all  $s \geq 0$  and

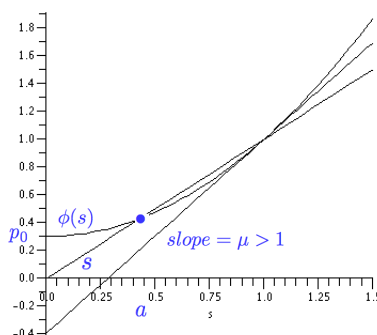
$$\phi''(s) > 0$$

meaning the function is concave up for all  $s > 0$  if at least one of the numbers  $p_2, p_3, \dots$  is positive. For example, this must be true if  $\mathbb{E}(K) = 1$  since  $K$  is sometimes zero:  $p_0 = \mathbb{P}(K = 0) > 0$ , it must also sometimes be greater than 1 to have an average of 1.

**Lemma 2.3.**  *$a$  is the smallest positive real number so that  $a = \phi(a)$ .*

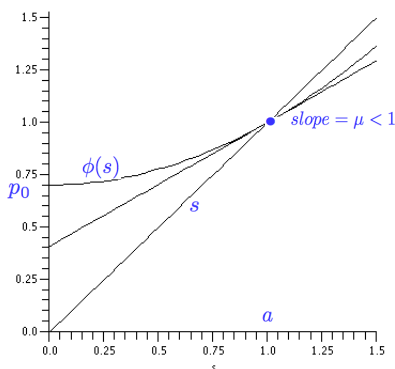
I will give the proof later. First, I want to show what this means.

Case 1. Suppose that  $\mu > 1$ . In that case, the slope of the curve  $y = \phi(s)$  at  $s = 1$  is greater than 1. Since the  $y$ -intercept is  $\phi(0) = p_0 > 0$ ,  $\phi(1) = 1$  and  $\phi$  is concave up, the graph must look like this:



The curves  $y = \phi(s)$  and  $y = s$  must meet at some point  $s = a = \phi(a)$  where  $0 < a < 1$ . So, there is a nonzero probability of extinction and a nonzero probability of indefinite survival. It could go either way. Someone once told me “May you have a thousand sons.” Even if people have, on the average, a thousand children, if there is a chance of death then there is a chance of extinction of the species.

Case 2. Suppose  $\mu < 1$ . In this case the slope is less than 1 at  $s = 1$  and the graph looks like this:

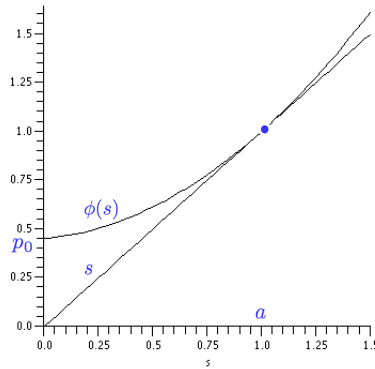


So,  $a = 1$ . This implies that

$$\mathbb{P}(\text{extinction} \mid X_0 = k) = a^k = 1.$$

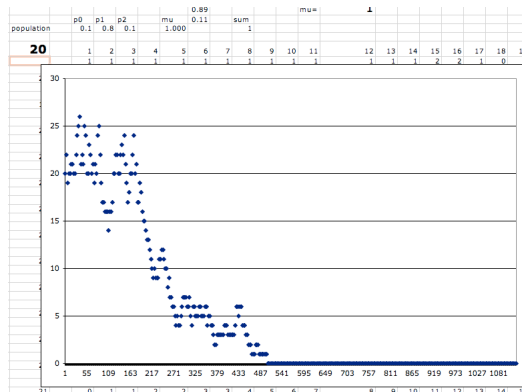
No matter how large is the population, if the number of children is, on the average, smaller than the number of parents, then the population will almost surely reach extinction. This is not surprising. It is the next case which is really interesting.

Case 3. Suppose  $\mu = 1$ . In this case the picture is:

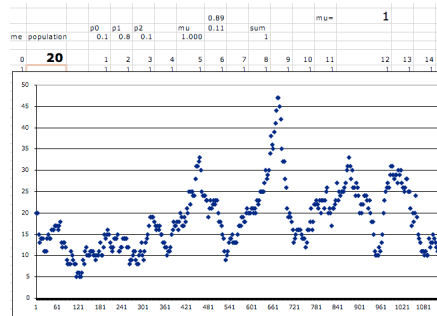


The curves  $y = \phi(s)$  and  $y = s$  must be touching at  $s = 1$ . This is the only point where these two curves meet. So,  $a = 1$  is the only solution of  $a = \phi(a)$ . So, even if the number of children is, on average, equal to the number of parents, the species will almost surely die out.

2.2.4. *computer simulations.* Using a random number generator, we looked at some examples of what might happen. When  $\mu = 1$ , the population sometimes dies out right away:

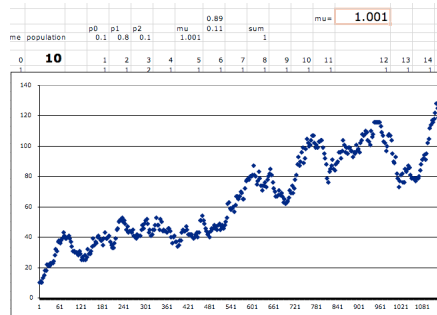


And sometimes it appears to be surviving:



However, according to the theory, the population will almost surely die out when  $\mu = 1$ . It just does not tell us how long it will take. We will discuss this later.

We did one more simulation with  $\mu = 1.001$ . In this case  $a \approx 99\%$ .



When  $X_0 = 10$  this makes the extinction probability  $a^{10} \approx 90\%$ . But 10% of the time, the species will survive indefinitely. This looks like one of these cases. After 375 generations, the population has reached  $X_{375} = 120$ . So, the extinction probability is now only

$$a^{120} \approx .99^{120} \approx 30\%.$$

2.2.5. *proof of extinction lemma.* The proof of Lemma 2.3 is just like the proof of the lemma I did on Wednesday. It goes like this. Suppose that  $\hat{a}$  is the smallest positive solution of the equation

$$\hat{a} = \phi(\hat{a}).$$

Then, to show that  $a = \hat{a}$ , it suffices to show that  $a \leq \hat{a}$ . But  $a = \lim_{n \rightarrow \infty} a_n$  where

$$a_n := \mathbb{P}(X_n = 0 \mid X_0 = 1).$$

So, it is enough to show (ETS), that

$$a_n \leq \hat{a}.$$

The proof of this is by induction on  $n$ .

If  $n = 0$  then  $a_0 = 0$ . Since  $\hat{a} > 0$ ,  $a_0 = 0 < \hat{a}$ . So, the statement holds for  $n = 0$ . Suppose the statement holds for  $n$ . Then we have to show that it holds for  $n + 1$ . But,

$$\begin{aligned} a_{n+1} &= \mathbb{P}(X_{n+1} = 0 \mid X_0 = 1) = \sum_{k=0}^{\infty} \mathbb{P}(X_{n+1} = 0 \mid X_1 = k) \mathbb{P}(X_1 = k \mid X_0 = 1) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(X_{n+1} = 0 \mid X_1 = 1)^k p_k = \sum_{k=0}^{\infty} a_n^k p_k \end{aligned}$$

By the assumption that  $a_n \leq \hat{a}$ ,

$$\sum_{k=0}^{\infty} a_n^k p_k \leq \sum_{k=0}^{\infty} \hat{a}^k p_k = \phi(\hat{a}) = \hat{a}.$$

So,  $a_{n+1} \leq \hat{a}$ . By induction this implies that  $a_n \leq \hat{a}$  for all  $n$ . But then,

$$a = \lim_{n \rightarrow \infty} a_n \leq \hat{a}.$$

**2.3. Random walk on  $\mathbb{Z}^d$ .** We talked about random walks on the  $d$ -dimensional integer lattice  $\mathbb{Z}^d$  and discussed whether they are recurrent or transient.

We take

$$S = \mathbb{Z}^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i \in \mathbb{Z}\}.$$

This is the integer lattice in  $\mathbb{R}^d$ . Each point  $x \in \mathbb{Z}^d$  has exactly  $2d$  neighbors  $y$  which are the integer points which are exactly 1 unit away from  $x$ . This is the easiest to see if  $x = (0, 0, \dots, 0)$  is the origin. Then the neighbors are:

$$\begin{aligned} &(\pm 1, 0, \dots, 0), \\ &(0, \pm 1, 0, \dots, 0), \dots, \\ &(0, \dots, 0, \pm 1). \end{aligned}$$

There are 2 in each of the  $d$  directions.

If the point moves to a neighbor with equal probability then we get:

$$p(x, y) = \mathbb{P}(X_{n+1} = y \mid X_n = x) = \begin{cases} \frac{1}{2d} & \text{if } \|x - y\| = 1 \\ 0 & \text{otherwise} \end{cases}$$

**2.3.1. recurrent or transient?** I reminded you that a Markov chain is *irreducible* if it has only one communication class.

**Definition 2.4.** An irreducible countably infinite Markov chain is called *recurrent* if, for any starting point  $X_0 = x$ ,

- (1) you almost surely return to  $x$ .
- (2)  $\Rightarrow$  You a.s. return to  $x$  an infinite number of times.
- (3)  $\Rightarrow$  You a.s. go to every other state  $z \in S$  an infinite number of times.

The conclusion is: If a Markov chain is recurrent then you will, with probability one, visit every state in the system an infinite number of times. This is the definition.

To figure out if this is true, we make it into an equation. You return to  $x$  an infinite number of times if the expected number of visits to  $x$  is infinity:

$$\mathbb{E}(\text{returns to } x \mid X_0 = x) = \infty.$$

**2.3.2. the case  $d = 1$ .** In the one dimensional case, we are talking about a random walk on the integers  $S = \mathbb{Z}$ . At each step we go either left or right with probability  $\frac{1}{2}$ . This is periodic with period 2. We need an even number of steps to get back to the starting point.

Let's say the starting point is 0. Then, we want to calculate  $p_{2n}(0, 0)$ . But, to return to 0 after  $2n$  moves, we need to take exactly  $n$  moves to the right and  $n$  moves to the left. There are

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

ways to do this. Each way, say  $RRLLRL$ , has probability  $(1/2)^{2n}$ . So, the total probability is

$$p_{2n}(0, 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{n!n!} \left(\frac{1}{2}\right)^{2n}$$

We used Stirling's formula:

$$n! \sim e^{-n} n^n \sqrt{2\pi n}$$

where  $\sim$  means the ratio of the two sides converges to 1 as  $n \rightarrow \infty$ . (In fact, you can check with a computer that this ratio starts at 1.084437 and slowly decreases to 1 as  $n$  goes to  $\infty$ ). So,

$$(2n)! \sim e^{2n} (2n)^{2n} \sqrt{4\pi n}$$

and

$$\frac{(2n)!}{n!n!} \sim \frac{e^{2n} (2n)^{2n} \sqrt{4\pi n}}{e^{2n} n^{2n} 2\pi n} = \frac{2^{2n}}{\sqrt{\pi n}}$$

which means that

$$p_{2n}(0, 0) \sim \frac{1}{\sqrt{\pi n}}$$

The expected number of return visits to 0 is

$$\sum_{n=1}^{\infty} p_{2n}(0, 0) \approx \sum \frac{1}{\sqrt{\pi n}} > \sum \frac{1}{\pi n} = \infty.$$

So, 0 is recurrent in  $\mathbb{Z}$ . Since there is only one communication class, the entire Markov chain is recurrent by definition.

**2.3.3. Higher dimensions.** In the planar lattice  $\mathbb{Z}^2$ , both coordinates must be 0 at the same time in order for the particle to return to the origin. Therefore,

$$p_{2n}(0, 0) \sim \frac{1}{\pi n}$$

and

$$\sum_{n=1}^{\infty} p_{2n}(0, 0) \approx \sum \frac{1}{\pi n} = \infty$$

and  $\mathbb{Z}^2$  is also recurrent.

When  $d > 2$  we get

$$p_{2n}(0, 0) \sim \frac{1}{(\pi n)^{d/2}}$$

and

$$\sum_{n=1}^{\infty} p_{2n}(0, 0) \approx \sum \frac{1}{(\pi n)^{d/2}}$$

which converges by the integral test. So, the expected number of visits is finite and  $\mathbb{Z}^3, \mathbb{Z}^4$ , etc are transient.

2.3.4. *criterion for transient Markov chains.* Next I explained how you can tell when a Markov chain is transient. I used the following function. Take a fixed state  $z \in S$ .

Let  $\alpha(x)$  be the probability that you go from  $x$  to  $z$ . I.e., you start at  $x$  and see whether you are ever in state  $z$ :

$$\alpha(x) := \mathbb{P}(X_n = z \text{ for some } n \geq 0 \mid X_0 = x)$$

Since the condition includes the case  $n = 0$ , we have

- (1)  $\alpha(z) = 1$ . Also,
- (2)  $0 \leq \alpha(x) \leq 1$  (since  $\alpha(x)$  is a probability). And:
- (3) If  $x \neq z$  then

$$\alpha(x) = \sum_{y \in S} p(x, y) \alpha(y)$$

(To get from  $x \neq z$  to  $z$  you take one step to  $y$  and then go from  $y$  to  $z$ .)

This boxed equation, when written in matrix form is:

$$\alpha = P\alpha.$$

In other words,  $\alpha$  is a right eigenvector of  $P$  with eigenvalue 1. But we have another eigenvector with the same eigenvalue:

$$\beta = P\beta$$

where

$$\beta = (1, 1, 1, \dots)$$

(Actually  $\beta$  is a column vector.) There is more than one eigenvector with eigenvalue 1. The following theorem tells us which vector we want.

**Theorem 2.5.** *The function  $\alpha$  is the smallest function satisfying the three conditions above. In other words, if  $\hat{\alpha}$  is any function satisfying these three conditions, then*

$$\alpha(x) \leq \hat{\alpha}(x)$$

for all states  $x$ .

The next theorem tells us why this is important.

**Theorem 2.6.** *The Markov chain is recurrent if  $\alpha = \beta$ , i.e.,  $\alpha(x) = 1$  for all  $x \in S$ .*

*The Markov chain is transient if  $\alpha < \beta$ , i.e.,  $\alpha(x) < 1$  for some state  $x \in S$ .*

This second theorem is just the definition of recurrent!  $\alpha(x) = 1$  means we almost surely go from  $x$  to  $z$ . This is the same as saying the chain is recurrent.

The proof of the first theorem is the same as the proof of the extinction lemma.

*Proof.* First I pointed out that

$$\alpha(x) = \lim_{n \rightarrow \infty} \alpha_n(x)$$

where

$$\alpha_n(x) := \mathbb{P}(X_0 = z \text{ or } X_1 = z \text{ or } \cdots \text{ or } X_n = z \mid X_0 = x).$$

So, it is enough to show that

$$\alpha_n(x) \leq \hat{\alpha}(x)$$

for all  $n \geq 0$ . I proved this by induction on  $n$ .

If  $n = 0$  then

$$\alpha_0(x) = \mathbb{P}(X_0 = z \mid X_0 = x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{if } x \neq z \end{cases}$$

But  $\hat{\alpha}$  is a function that satisfies the three conditions and condition (1) says that  $\hat{\alpha}(z) = 1$ .

$$\alpha_0(z) = 1 \leq \hat{\alpha}(z).$$

Condition (2) says that  $\hat{\alpha}(x) \geq 0$  for all  $x$ . So,

$$\alpha_0(x) = 0 \leq \hat{\alpha}(x) \text{ for } x \neq z.$$

So, the statement holds for  $n = 0$ .

Suppose the statement holds for  $n$  (i.e.,  $\alpha_n(x) \leq \hat{\alpha}(x)$  for all  $x \in S$ ). Then,

$$\alpha_{n+1}(x) = \sum_{y \in S} p(x, y) \alpha_n(y)$$

By induction, this is

$$\leq \sum_{y \in S} p(x, y) \hat{\alpha}(y)$$

which is equal to  $\hat{\alpha}(x)$  by condition (3). So,

$$\alpha_{n+1}(x) \leq \hat{\alpha}(x)$$

for all  $x$ . By induction the statement holds for all  $n$ . Taking limits we get that

$$\alpha(x) = \lim_{n \rightarrow \infty} \alpha_n(x) \leq \hat{\alpha}(x).$$

□

In the case of a random walk on  $S = \mathbb{Z}$  with unequal probabilities of going right and left, we can find the function  $\alpha$  and prove that the Markov chain is transient.

**2.4. Transient, recurrent and null recurrent.** [Guest lecture by Alan Haynes. Notes by Connie, Peter, Sam, Xioudi, Zach.] Alan Haynes reviewed the difference between transient and recurrent and defined the notion of null recurrence.

2.4.1. *transient and recurrent.* Suppose that  $\{X_n\}$  is an irreducible aperiodic Markov chain. Fix  $z \in S$  and define the function  $\alpha : S \rightarrow [0, 1]$  by

$$\alpha(x) = \mathbb{P}(X_n = z \text{ for some } n \geq 0 \mid X_0 = x)$$

Facts:

- (1)  $\alpha(x) \in [0, 1]$
- (2)  $\alpha(z) = 1$
- (3) If  $x \neq z$ , then

$$\alpha(x) = \sum_{y \in S} p(x, y) \alpha(y)$$

$\Rightarrow \alpha$  is a right eigenvector of  $P = (p(x, y))_{x, y \in S}$  (the probability transition matrix).

There are two possibilities:  $X_n$  is either transient or recurrent.

- (4) Transient  $\Leftrightarrow$  the probability of returning to  $z$  (or any other point) an infinite number of times is 0.

$$\Leftrightarrow \alpha(x) < 1 \text{ for some } x \in S$$

$$\Leftrightarrow \inf_{x \in S} \alpha(x) = 0$$

- (5) Recurrent  $\Leftrightarrow$  the probability of returning to every point an infinite number of times is 1.

$$\Leftrightarrow \alpha(x) = 1 \quad \forall x \in S$$

Why is it true that  $\alpha(x) < 1$  implies  $\inf \alpha(x) = 0$ ? The proof is by contradiction. Suppose not. Then there is some  $\epsilon > 0$  so that  $\alpha(x) \geq \epsilon$  for all  $x \in S$ . This implies that, no matter where you are, you have a probability of at least  $\epsilon$  of returning to  $z$ . So, with probability one, you must return to  $z$  an infinite number of times making the chain recurrent.

**Theorem 2.7.** *If  $\hat{\alpha} : S \rightarrow [0, 1]$  satisfies*

$$\hat{\alpha}(x) = \sum_{y \in S} p(x, y) \hat{\alpha}(y) \text{ for all } x \in S \setminus z$$

*then  $\alpha(x) \leq \hat{\alpha}(x)$  for all  $x \in S$ .*

To show that a Markov chain is transient, you should find a solution  $\hat{\alpha}(x)$  of the equation so that  $\hat{\alpha}(x) < 1$  at some point  $x \in S$ .

Example: Take the random walk on  $\{0, 1, 2, \dots\}$  with partially reflecting wall at 0. This means that, at 0, you either stay at 0 or go to 1. Choose some  $p$  in the open interval  $(0, 1)$  and let  $q = 1 - p$ . If  $n \geq 1$ , the transition probabilities are given by

$$p(n, n-1) = p \quad (\text{going to the left})$$

$$p(n, n+1) = q \quad (\text{going to the right})$$

At  $n = 0$  you can't go left. The probabilities are given by

$$p(0, 0) = p, \quad p(0, 1) = q.$$

Question: Is this transient or recurrent? We need to choose a point  $z$  and use the theorem.

Choose  $z = 0$ . Then find a function  $\hat{\alpha}$  that satisfies the equation in the theorem:

$$\hat{\alpha}(n) = \sum_{m \in \mathbb{N}} p(n, m) \hat{\alpha}(m) \quad \text{for } n \geq 1$$

Since  $p(n, m) = 0$  unless  $m = n+1, n-1$ , we can simplify this to:

$$\hat{\alpha}(n) = p\hat{\alpha}(n-1) + q\hat{\alpha}(n+1)$$

This is a homogeneous linear difference equation with constant coefficients. So,

$$\hat{\alpha}(n) = C_1 r^n + C_2 s^n$$

where  $C_1, C_2 \in \mathbb{R}$  and  $r, s$  are the roots of the equation

$$x = p + qx^2$$

If you use the quadratic equation you get:

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q}, \quad s = \frac{1 + \sqrt{1 - 4pq}}{2q}$$

If you use the fact that  $p + q = 1$  you get the solution:

$$r, s = \frac{p}{q}, 1 \quad (\text{The smaller number is } r.)$$

To get this you need to factor the equation:

$$x = (p + q)x = p + qx^2$$

$$p(x-1) = q(x^2 - x)$$

This is true if  $x = 1$  or, when  $x \neq 1$  you can factor out the term  $x - 1$  to get:

$$p = qx.$$

Since the actual probability is  $\alpha(n) \leq \hat{\alpha}(n)$ , we want  $\hat{\alpha}(n) < 1$ . This happens when one of the roots  $r, s$  is less than 1.

Case 1:  $p = \frac{1}{2}$ . Then  $q = \frac{1}{2}$ . The roots are  $r = s = 1$ . Since

$$\hat{\alpha}(0) = C_1 r^0 + C_2 s^0 = 1$$

we have

$$\hat{\alpha}(n) = C_1 r^n + C_2 s^n = C_1 + C_2 = 1.$$

So, the only solution of the equation is  $\hat{\alpha}(n) = 1$ . This implies the chain is recurrent. (In fact, it is *null recurrent* as explained below.)

Case 2:  $p < \frac{1}{2}$ . Then  $p = \frac{1}{2} - \epsilon$ ,  $q = \frac{1}{2} + \epsilon$ ,  $pq = \frac{1}{4} - \epsilon^2$ ,  $1 - 4pq = 4\epsilon^2$ ,

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q} = \frac{1 - 2\epsilon}{1 + 2\epsilon} = \frac{p}{q} < 1.$$

Then, one of the solutions is:

$$\hat{\alpha}(n) = r^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

This implies that the chain is transient.

Note: We only need one  $\hat{\alpha}(n) < 1$  to prove transience.

Case 3:  $p > \frac{1}{2}$ . Then  $p = \frac{1}{2} + \epsilon$ ,  $q = \frac{1}{2} - \epsilon$ ,  $pq = \frac{1}{4} - \epsilon^2$ ,

$$r = \frac{1 - \sqrt{1 - 4pq}}{2q} = \frac{1 - 2\epsilon}{1 - 2\epsilon} = 1.$$

The other solution is

$$s = \frac{1 + \sqrt{1 - 4pq}}{2q} = \frac{1 + 2\epsilon}{1 - 2\epsilon} = \frac{p}{q} > 1.$$

This implies the chain is recurrent. (In fact, it is *positive recurrent*.)

2.4.2. *null recurrence.*

**Definition 2.8.** An irreducible aperiodic chain  $\{X_n\}$  is called *null recurrent* if it is recurrent and

$$\lim_{n \rightarrow \infty} p_n(x, y) = 0 \quad \forall x, y \in S.$$

If  $\{X_n\}$  is recurrent but not null recurrent then it is called *positive recurrent*.

How do you tell the difference?

**Theorem 2.9.** *If  $\{X_n\}$  is positive recurrent then, for every  $x, y \in S$ ,*

$$\lim_{n \rightarrow \infty} p_n(x, y) = \pi(y) > 0$$

where  $\pi : S \rightarrow [0, 1]$  is the invariant probability distribution.

Recall the definition of invariant distribution:

- (1) (probability distribution)  $\sum_{x \in S} \pi(x) = 1$
- (2) (invariant)  $\pi P = \pi :$

$$\pi(y) = \sum_{x \in S} \pi(x) p(x, y).$$

I.e.,  $\pi$  is a left eigenvector of  $P$ .

**Corollary 2.10.**  *$\{X_n\}$  is positive recurrent if and only if it has an invariant distribution.*

## HOMEWORK 2: COUNTABLE MARKOV CHAINS

Four problems due Tuesday, March 4 by 5pm:

From the book: Chap 2, #1,2,14,16.

These are the hints given in class last week:

**Queueing model.** In this problem,

$X_n = \#$  customers waiting in line at time  $n$ .

$p =$  probability that a new customer arrives

$q =$  probability that the customer at the front of the line leaves.

$$S = \{0, 1, 2, 3, \dots\}$$

The probability transition matrix is given by:

$$p(x, x + 1) = p(1 - q) \text{ if } x \geq 1$$

since a customer must enter the line and the one at the front must stay in order for the line to get longer. Similarly,

$$p(x, x - 1) = q(1 - p) \text{ if } x \geq 1.$$

The only other possibility is that the line remains the same:

$$p(x, x) = 1 - p(1 - q) - q(1 - p)$$

If  $x = 0$  then the line can't get shorter. So

$$p(0, 1) = p, \quad p(0, 0) = 1 - p.$$

The question is: When is this transient, positive recurrent, null recurrent. The answer is intuitively obvious:

$p > q$  In this case, new customers are entering the line at a rate faster than they are being served. The line will get longer and longer. This is transient because each state (say line size 100) will occur only a few times and then never again.

$p < q$  Customers are being served faster than they enter the line. This line will shrink to zero on a regular basis. This is positive recurrent.

$p = q$  Customers come and go at the same rate. This line will just randomly fluctuate between larger and smaller lengths. Since the fluctuations get really large it hits zero just by dumb chance. It also takes longer and longer for each cycle. This is null recurrent. However, this conclusion is not so obvious.

Your job is to verify this by finding  $\alpha$  in the transient case and  $\pi$  in the positive recurrent case.

Transient case. You should take  $z = 0$  and find a function  $\hat{\alpha}$  so that

$$\hat{\alpha}(x) = \sum_{y=0}^{\infty} p(x, y)\hat{\alpha}(y) \text{ if } x \neq 0$$

If  $\hat{\alpha}(z) = 1$ ,  $0 \leq \hat{\alpha}(x) \leq 1$  and  $\hat{\alpha}(x) < 1$  for some  $x$  then the Markov chain is transient. ( $\hat{\alpha}$  is a solution of the equation,  $\alpha$  is the probability of returning to  $z = 0$ . If  $\hat{\alpha}(x) < 1$  then  $\inf \alpha(x) = 0$ .)

Positive recurrent In this case you need to find a function  $\pi(x) \geq 0$  so that

$$\pi(y) = \sum_{x=0}^{\infty} \pi(x)p(x, y)$$

and

$$\sum \pi(x) < \infty$$

Then you can divide by this sum to get the invariant distribution.

Null recurrent When  $\alpha, \pi$  do not exist, the chain is null recurrent.

In problem 2, I explained that the Markov chain is (+)-recurrent if and only if  $T$ , the expected return time to zero is finite:

$$\mathbb{E}(T) = \sum_{n=1}^{\infty} n\mathbb{P}(T = n)$$

But  $\mathbb{P}(T = n) = p_n$ .